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Edited by

M. Schlichenmaier
University of Mannheim
Mannheim, Germany

A. Strasburger
University of Białystok
Białystok, Poland

S. Twareque Ali
Concordia University
Montréal, Quebec, Canada

A. Odzijewicz
University of Białystok
Białystok, Poland

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ROTATING DUST MODELS IN RELATIVITY

Andrzej Krasiński
N. Copernicus Astronomical Center and College of Science of PAN, Warsaw

For rotating dust with a 3-dimensional symmetry group all possible metric forms can be classified and, within each class, explicitly written out. With respect to the structures of the groups, this is just the Bianchi classification, but with all orientations of the orbits taken into account. This result follows from the formalism introduced by Plebański. This paper is a brief overview of results that are published elsewhere.

1. Introduction and summary

There exist unsolved problems even in classical relativity. One of them is finding an exact solution of Einstein's equations that could describe an expanding and rotating Universe. What is it needed for? A physicist's approach to Nature is to first calculate the magnitude of an effect from theory, and then to measure it — in order, among other things, to test the theory. The same principle should apply to cosmology (although not all astronomers share this view; some of them tend to just *know* some things without verification). However, in astronomy, unlike in physics, we cannot carry out active experiments; we can only do passive observations. In order to calculate what observable effects rotation of the Universe might have, it is necessary to have at least one acceptable model of the Universe that allows for rotation. Such a model is still lacking. Existing solutions of Einstein's equations with rotating sources are either stationary from the beginning or become static in the limit of zero rotation. The approach presented here aims at obtaining a solution that would be free of these problems. So far, the approach has not resulted in a success, but it seems promising.

The theorem of Darboux (see Section 2) allows one to introduce invariantly defined coordinates in which the velocity field of a fluid acquires a preferred form. It is assumed in addition that the fluid moves with zero acceleration and nonzero rotation. These assumptions result in a simplification of the metric tensor and in limitations imposed on the Killing vectors, if any exist. A Killing field k^α may be spanned on velocity u^α and rotation w^α or may be linearly independent of u^α and w^α. This gives rise to a classification of possible symmetries in the presence of rotating matter.

When there exist three linearly independent Killing fields, the classification described above gives rise to a complete classification of all such metric forms. With respect to the algebras of the symmetry groups, this is just the Bianchi classification, but with all possible orientations of the orbits in the manifold taken into account (i.e. they may be time-like, space-like or null).

For later reference, let us note the relation [27] between the Bianchi classification and the non-rotating cosmological models of Friedmann that are commonly used by astronomers as the standard models of our real Universe. The $k < 0$ Friedmann model is contained in Bianchi types V and VII_h, the $k = 0$ Friedmann model is contained in Bianchi types I and VII_0, the $k > 0$ model is contained in the Bianchi type IX.

In every case that emerges, the commutation relations of the symmetry algebra have been solved, resulting in explicit formulae for the Killing fields, and then the Killing equations have been solved, resulting in the formulae for the metric tensors compatible with the symmetry group considered. The degree of success in solving the Einstein equations varied from case to case. In most cases, no progress was made. In a few cases the Einstein equations have been partly integrated and reduced to a simpler set. Several solutions known earlier were identified in the present scheme (those by Lanczos [1], Gödel [2], Maitra [3], Ellis [4], King [5], Vishveshwara and Winicour [6] and a few solutions with rotating charged dust, see below). In this note only the most important results will be described, the full results are published elsewhere [24–26].

In Section 2 the Darboux theorem and the associated classification of first-order differential forms are introduced. In Section 3, the classification is applied to geodesic vector fields with rotation. When the vector field is the velocity field of a fluid, a class of preferred coordinates results (which shall be termed "Plebański coordinates" [7]). In Section 4 it is shown that each Killing vector field that might possibly exist in a space-time with a geodesic and rotating fluid source is determined by two functions of two variables. If the Killing field is not spanned on velocity and rotation, then the Plebański coordinates may be adapted to it so that it acquires the form $k^\alpha = \delta^\alpha_1$.

In Section 5, the consideration of Section 4 is applied to the case of three Killing vector fields existing on a manifold. When all three of them are spanned on u^α and w^α, the group becomes two-dimensional and this case is not considered here. When two of them are spanned on u^α and w^α while the third one is not, two cases arise. In one of them (Bianchi type II), the Einstein equations reduce to a single ordinary differential equation of third order (of second order when $\Lambda = 0$). In the other case (Bianchi type I), the Einstein equations are reduced to a set of first-order equations. The solutions of Lanczos [1] and of Gödel [2] emerge as special cases of both these classes.

The remaining cases are sketched only briefly. Section 6 contains the description of the case when two of the Killing fields are linearly independent of u^α and w^α, Section 7 — of the case when all three Killing fields are linearly independent of u^α and w^α. With the increasing number of Killing vectors that are linearly independent of u^α and w^α, the number of subcases requiring separate treatment increases, and the Einstein equations become progressively more complicated. In Section 8 the most promising case is briefly described.

2. The classification of differential forms of first order and the Darboux theorem

Definition. *Let q be a differential form of first order.*
If $Q_{2l} := dq \wedge \ldots \wedge dq$ (multiplied l times) $\neq 0$, but $q \wedge Q_{2l} = 0$, then q is said to be of class $2l$.
If $Q_{2l+1} := q \wedge Q_{2l} \neq 0$, but $dQ_{2l+1} \equiv dq \wedge Q_{2l} = 0$, then q is said to be of class $(2l+1)$.

Then the following holds.

Theorem. (Darboux) *The form q is of class $2l$ if and only if there exists a set of $2l$ independent functions $(\xi_1, \ldots, \xi_l, \eta_1, \ldots, \eta_l)$ such that*

$$q = \eta_1 d\xi_1 + \eta_2 d\xi_2 + \cdots + \eta_l d\xi_l. \tag{2.1}$$

The form q is of class $(2l+1)$ if and only if there exists a set of $(2l+1)$ independent functions $(\tau, \xi_1, \ldots, \xi_l, \eta_1, \ldots, \eta_l)$ such that

$$q = d\tau + \eta_1 d\xi_1 + \eta_2 d\xi_2 + \cdots + \eta_l d\xi_l. \tag{2.2}$$

(See [9] for a proof.)

The Darboux theorem implies that in a four-dimensional space-time V_4 the most general differential form of first order can be represented as

$$q = \sigma d\tau + \eta d\xi, \tag{2.3}$$

where σ, τ, η and ξ are scalar functions on V_4.

Any vector field u^α on V_4 defines the following form of first order

$$q_u := u_\alpha dx^\alpha. \tag{2.4}$$

According to (2.3), in the most general case there exist scalar functions σ, τ, η and ξ such that

$$u_\alpha = \sigma \tau,_\alpha + \eta \xi,_\alpha . \tag{2.5}$$

In general, the four functions are independent, i.e.

$$\frac{\partial(\sigma, \tau, \eta, \xi)}{\partial(x^0, x^1, x^2, x^3)} \neq 0. \tag{2.6}$$

In that case, they can be chosen as coordinates in the space-time.

3. Geodesically moving fluids

To any time-like vector field u_α normalized to unity (so that $u_\alpha u^\alpha = 1$) the decomposition described in [10, 11] may be applied

$$u_{\alpha;\beta} = \dot{u}_\alpha u_\beta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}, \tag{3.1}$$

where $\dot{u}^\alpha$, θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are, respectively, the acceleration, expansion, shear and rotation. In the signature $(+---)$ used here, the projection tensor $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta. \tag{3.2}$$

The following identities hold

$$\dot{u}_\alpha u^\alpha = 0, \qquad \sigma_{\alpha\beta}u^\beta = \omega_{\alpha\beta}u^\beta = 0. \tag{3.3}$$

We shall assume from now on that u_α is the velocity field of a fluid and that $\dot{u}_\alpha = 0$, i.e. that the particles of the fluid move on geodesics. Then, from (2.5) we have

$$\omega_{\alpha\beta} = \frac{1}{2}(\sigma,_\beta \tau,_\alpha - \sigma,_\alpha \tau,_\beta + \eta,_\beta \xi,_\alpha - \eta,_\alpha \xi,_\beta), \tag{3.4}$$

and from (3.3) we have

$$(u^\beta\sigma,_\beta)\tau,_\alpha - (u^\beta\tau,_\beta)\sigma,_\alpha + (u^\beta\eta,_\beta)\xi,_\alpha - (u^\beta\xi,_\beta)\eta,_\alpha = 0. \tag{3.5}$$

It is easy to see that, in virtue of (3.5), the form (2.4) cannot be of class 4. Hence, for a geodesically moving fluid the form (2.4) is of class at most 3, i.e. at most 3 independent functions τ, η, ξ exist such that

$$u_\alpha = \tau,_\alpha + \eta\xi,_\alpha . \tag{3.6}$$

The functions $\{\tau, \xi, \eta\}$ in (3.6) must then obey $u^\alpha\xi,_\alpha = u^\alpha\eta,_\alpha = 0$ (from (3.3)) and are determined up to the following transformations

$$\xi = F(\xi', \eta'), \qquad \eta = G(\xi', \eta'), \qquad \tau = \tau' - S(\xi', \eta'), \tag{3.7}$$

where the functions F and G must obey the equation

$$F,_{\xi'} G,_{\eta'} - F,_{\eta'} G,_{\xi'} = 1, \tag{3.8}$$

and then S is determined by

$$S,_{\xi'} = GF,_{\xi'} - \eta', \qquad S,_{\eta'} = GF,_{\eta'} . \tag{3.9}$$

Let us now assume that the number of particles of the fluid is conserved, i.e.

$$(\sqrt{-g}nu^\alpha),_\alpha = 0, \tag{3.10}$$

where g is the determinant of the metric tensor and n is the particle number density. This equation is a necessary and sufficient condition for the existence of a function ζ such that:

$$\sqrt{-g}nu^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\xi,_\beta\,\eta,_\gamma\,\zeta,_\delta\,. \tag{3.11}$$

Note that Eqs. (3.3) and (3.6) imply that

$$u^\alpha\tau,_\alpha = 1, \tag{3.12}$$

and then Eq. (3.11) implies that

$$\varepsilon^{\alpha\beta\gamma\delta}\tau,_\alpha\,\xi,_\beta\,\eta,_\gamma\,\zeta,_\delta \equiv \frac{\partial(\tau,\eta,\xi,\zeta)}{\partial(x^0,x^1,x^2,x^3)} = \sqrt{-g}n \neq 0, \tag{3.13}$$

$$u^\alpha\zeta,_\alpha = 0. \tag{3.14}$$

The function ζ is determined by (3.11) up to the transformations

$$\zeta = \zeta' + T(\xi',\eta'), \tag{3.15}$$

where T is an arbitrary function. Eq. (3.13) certifies that $\{\tau,\xi,\eta,\zeta\} = \{x^0, x^1, x^2, x^3\} := \{t,x,y,z\}$ can be used as coordinates in the space-time. With such a choice, Eq. (3.6) implies

$$u_0 = 1, \qquad u_1 = y, \qquad u_2 = u_3 = 0. \tag{3.16}$$

We will use these coordinates throughout the remaining part of the paper and call them "Plebański coordinates". Eq. (3.13) implies now

$$g = -n^{-2}, \tag{3.17}$$

and Eq. (3.11) implies

$$u^\alpha = \delta^\alpha_0, \tag{3.18}$$

i.e. the Plebański coordinates are co-moving. The rotation vector defined by

$$w^\alpha = \frac{1}{2\sqrt{-g}}\varepsilon^{\alpha\beta\gamma\delta}u_\beta u_{\gamma,\delta} \tag{3.19}$$

assumes the form

$$w^\alpha = \frac{1}{2}n\delta^\alpha_3. \tag{3.20}$$

Eqs. (3.16) and (3.18) imply that

$$g_{00} = 1, \qquad g_{01} = y, \qquad g_{02} = g_{03} = 0, \tag{3.21}$$

and also that the only nonvanishing components of the rotation tensor are

$$\omega_{12} = -\omega_{21} = 1/2. \tag{3.22}$$

If we now assume that the fluid is perfect, then we conclude from the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ that either $\omega = 0$ or $p = \text{const}$ (see also [12]). This means that a geodesic perfect fluid can be rotating only if it is in fact dust; the constant p can be reinterpreted as the cosmological constant. For dust, the energy-density obeys the conservation equation $(\sqrt{-g}\epsilon u^\alpha)_{,\alpha} = 0$ and Eq. (3.10) need not be assumed separately. A more detailed exposition of the same material can be found in [8].

Note that rotating dust is not the only example to which this approach may be applied. In several papers, rotating charged dust was considered under the additional assumptions that all charges are attached to the dust particles, that the only current is the one created by the flow of dust and that the Lorentz force acting on the dust particles, $F^\mu{}_\nu u^\nu$, is zero (i.e. that the electric and magnetic fields are such that they cancel each other's influence on the charged dust particles). Under these assumptions the dust particles move on geodesics and the formalism of this section applies. Such solutions of the Einstein–Maxwell equations were considered in [13–20]; they will be mentioned in Section 5.

4. The Killing vector fields compatible with rotation

We shall assume that the symmetries of the space-time (if any exist) are inherited by the source, i.e. that if the Lie derivative of the metric tensor $g_{\alpha\beta}$ along the vector field k^α is zero, $\pounds_k g_{\alpha\beta} = 0$, then the velocity field and the particle number density are also invariant: $\pounds_k u^\alpha = 0 = \pounds_k n$. (For a pure perfect fluid source the inheritance is guaranteed.) It follows that the rotation tensor must also be invariant, $\pounds_k \omega_{\alpha\beta} = 0$. All these equations imply that

$$k^0 = C + \phi - y\phi_{,y}\,, \qquad k^1 = \phi_{,y}\,, \qquad k^2 = -\phi_{,x}\,, \qquad k^3 = \lambda, \tag{4.1}$$

where $\phi(x,y)$ and $\lambda(x,y)$ are arbitrary functions and C is an arbitrary constant. If there are no symmetries, then $\phi = \lambda = C = 0$. However, if any symmetries are present, then the Killing vector fields must have the form (4.1).

Suppose that ϕ is not a constant, i.e. that a Killing vector field k^α exists that has a nonzero component in the x- or y-direction (in invariant terms this means that the vector field k^α is not spanned on the vector fields of velocity, u^α, and rotation, w^α). We can then, within the Plebański class defined in Section 3, adapt the coordinates to k^α in such a way that $k^{\alpha'} = \delta^{\alpha'}_1$, i.e. so that the metric becomes independent of x'. From Eqs. (3.7)–(3.9) and (3.15) the transformation functions are

$$t' = t - S(x,y), \qquad x' = F(x,y), \qquad y' = G(x,y), \qquad z' = z + T(x,y), \tag{4.2}$$

where T is arbitrary, while F, G and S obey

$$F_{,x}\,G_{,y} - F_{,y}\,G_{,x} = 1, \qquad S_{,x} = GF_{,x} - y, \qquad S_{,y} = GF_{,y}\,. \tag{4.3}$$

The condition $k^{\alpha'} = \delta_1^{\alpha'}$ then implies

$$G = \phi + C, \tag{4.4}$$

$$T_{,x}\, \phi_{,y} - T_{,y}\, \phi_{,x} = -\lambda. \tag{4.5}$$

Eqs. (4.3) and (4.5) simply define the accompanying F, S and T which are seen to exist always. Since ϕ was assumed nonconstant, the transformation is nonsingular, and results in $\phi = y$ in the new coordinates; the metric becomes independent of x after the transformation. This property is preserved by the transformations (4.2), but with F, G, S and T restricted now by

$$F = x + H(y), \qquad G = y, \qquad T = T(y), \qquad S = \int yH_{,y}\, dy + A, \tag{4.6}$$

where A is an arbitrary constant and H and T are arbitrary functions.

If three Killing vector fields are spanned on u^α and w^α, so that $\phi = \text{const}$ in (4.1) for each of them, i.e.

$$k_{(i)}^\alpha = C_i \delta_0^\alpha + \lambda_i(x, y)\delta_3^\alpha, \qquad i = 1, 2, 3, \tag{4.7}$$

then constants α_1, α_2 and α_3 exist such that $\alpha_1 k_{(1)} + \alpha_2 k_{(2)} + \alpha_3 k_{(3)} = 0$, i.e. the symmetry group is in fact two-dimensional. Hence, for a three-dimensional group at least one of the generators must be linearly independent of u^α and w^α at every point of the space-time region under consideration.

5. The case of two generators spanned on u^α and w^α

In this section we shall assume that exactly one generator, $k_{(1)}^\alpha$, is linearly independent of u^α and w^α, while the other two, $k_{(2)}^\alpha$ and $k_{(3)}^\alpha$, are of the form (4.7). In agreement with the result of Section 4, the Plebański coordinates can be adapted to $k_{(1)}^\alpha$ so that

$$k_{(1)}^\alpha = \delta_1^\alpha, \tag{5.1}$$

while

$$k_{(2)}^\alpha = C_2 \delta_0^\alpha + \lambda_2(x, y)\delta_3^\alpha, \qquad k_{(3)}^\alpha = C_3 \delta_0^\alpha + \lambda_3(x, y)\delta_3^\alpha, \tag{5.2}$$

and the coordinate transformations preserving (5.1) and (5.2) are (4.6). With no loss of generality we can assume that

$$C_2 \neq 0 = C_3, \tag{5.3}$$

because the Killing vector fields are determined up to linear combinations among them. This implies $\lambda_3 \neq 0$. The commutators of the Killing vectors are then

$$[k_{(1)}, k_{(2)}]^\alpha = (\lambda_{2,x}/\lambda_3) k_{(3)}^\alpha, \quad [k_{(2)}, k_{(3)}]^\alpha = 0, \quad [k_{(1)}, k_{(3)}]^\alpha = (\lambda_{3,x}/\lambda_3) k_{(3)}^\alpha. \tag{5.4}$$

The Killing vector fields will thus form a Lie algebra when

$$\lambda_{2,x} = b\lambda_3, \qquad \lambda_{3,x} = c\lambda_3, \tag{5.5}$$

where b and c are arbitrary constants. The cases $c \neq 0$ and $c = 0$ have to be considered separately. However, when $c \neq 0$, it follows that

$$\lambda_3 = \beta(y)e^{cx}, \qquad \lambda_2 = (b/c)\beta(y)e^{cx} + \alpha(y), \tag{5.6}$$

and so, with no loss of generality we can assume $b = 0$. The Einstein equations then imply that either $c = 0$ or there is no rotation. Since we are interested in rotating solutions only, this case need not be followed. We thus consider

Case I. $c = 0$, $b \neq 0$. Then

$$\lambda_3 = \beta(y), \qquad \lambda_2 = b\beta(y)x + \alpha(y). \tag{5.7}$$

The algebra of the Killing vector fields is of Bianchi type II when $b \neq 0$, and of Bianchi type I when $b = 0$.

In this case the Einstein equations and coordinate transformations can be used to simplify the metric form as follows:

$$ds^2 = (dt + Ydx)^2 - (Fdx)^2 - dy^2 - G^2(-btdx + dz)^2, \tag{5.8}$$

where the functions $Y(y)$ and $F(y)$ are defined as

$$F^2 = \Big(C - \frac{1}{2}Y + 2\frac{\Lambda}{B^2}\int G^2 dY\Big)G/G_{,Y}\,, \tag{5.9}$$

$$Y_{,y}G/F = B = \text{const}, \tag{5.10}$$

C is an arbitrary constant (we can assume $G_{,Y} \neq 0$ because $G_{,Y} = 0$ implies $b = 0$, and this will be considered separately), and $G(Y)$ is determined by the equation

$$-\frac{1}{4}b^2 GG_{,Y} + \frac{1}{2}(B/G)^2(C - \frac{1}{2}Y + 2\frac{\Lambda}{B^2}\int G^2 dY)^2(G_{,Y}/G - G_{,YY}/G_{,Y}) = 0, \tag{5.11}$$

see [24] for details. When $\Lambda = 0$ this becomes a second-order differential equation.

The formula for energy-density may be simplified to

$$(8\pi G/c^4)\epsilon = (B/G)^2 - (bG)^2 - 2\Lambda. \tag{5.12}$$

Note that the solutions considered here have a meaningful limit $b = 0$.

When $G = \text{const}$, Eqs. (5.9) and (5.11) no longer apply and one has to go back to the Einstein equations; the resulting metric is the Gödel solution.

When $G_{,Y} \neq 0 = b$, Eq. (5.11) implies $G = e^{DY+E}$, and this leads to the Lanczos solution (see [8]).

Case II. $b = c = 0$ in Eqs. (5.4)–(5.5).

The metric form has here more components (see again [24] for details).

$$ds^2 = (dt + Ydx)^2 - (Fdx)^2 - dy^2 - G^2[Adt - (-bt - k_{13})dx + dz]^2, \tag{5.13}$$

where the functions $Y(y), F(y), G(y), A(y)$ and $k_{13}(y)$ are determined by the following set of equations:

$$k_{13,y} = BF/G^3 - YA_{,y}\,, \tag{5.14}$$

$$Y_{,y} = (C - BA)F/G, \tag{5.15}$$

$$A_{,y} = (BY - D)/(FG^3), \tag{5.16}$$

$$FG_{,y} = \frac{1}{2}Bk_{13} + \frac{1}{2}BAY - \frac{1}{2}CY + 2\Lambda\int FGdy + H_0, \tag{5.17}$$

$$GF_{,y} = -\frac{1}{2}Bk_{13} - \frac{1}{2}DA - E + 2\Lambda\int FGdy + H_0, \tag{5.18}$$

where B, C, D and H_0 are arbitrary constants. Eqs. (5.14)–(5.18) are first integrals of the Einstein equations.

The functions $A(y)$ and $k_{13}(y)$ have invariant meaning: they are proportional to the scalar products of the Killing vectors:

$$A = -g_{\alpha\beta}k^{\alpha}_{(2)}k^{\beta}_{(3)}/G^2, \qquad k_{13} = -g_{\alpha\beta}k^{\alpha}_{(1)}k^{\beta}_{(3)}/G^2 \tag{5.19}$$

(note that $G^2 = -g_{\alpha\beta}k^{\alpha}_{(3)}k^{\beta}_{(3)}$, i.e. it is a scalar, too). Hence, $A = 0$ and $k_{13} = 0$ are invariant properties. Note that $A = 0$ implies, through (5.16), that either $Y =$ const (in which case there is no rotation) or $B = D = 0$. In the latter case, k_{13} = const and the coordinate transformation $z = z' - k_{13}x$ leads to $k_{13} = 0$ in the new coordinates. With $A = k_{13} = 0$, the Lanczos and Gödel models result from the Einstein equations as the only solutions.

In [13–20], charged dust solutions with zero Lorentz force were considered. Apart from the one in [13], they are cylindrically symmetric and stationary, and so they would emerge in this section, had we allowed charged dust as a source and considered the Einstein–Maxwell equations.

6. The case of one generator spanned on u^α and w^α

The number of cases that require separate treatment is larger here, and the equations are more complicated, so the results will be described only briefly.

One of the two generators that are not spanned on u^α and w^α may be given the form (5.1) by a coordinate transformation, the other will have the form (4.1). The one that is spanned on u^α and w^α will have the form (5.2). Hence, the generators are

$$k^\alpha_{(1)} = \delta^\alpha_1, \tag{6.1}$$

$$k^\alpha_{(2)} = (C_2 + \phi - y\phi_{,y})\delta^\alpha_0 + \phi_{,y}\delta^\alpha_1 - \phi_{,x}\delta^\alpha_2 + \lambda_2(x,y)\delta^\alpha_3, \tag{6.2}$$

$$k^\alpha_{(3)} = C_3\delta^\alpha_0 + \lambda_3(x,y)\delta^\alpha_3, \tag{6.3}$$

and the remaining freedom of coordinate transformations is given by Eqs. (4.2) and (4.6). The Killing fields (6.1)–(6.3) will form a Lie algebra when

$$[k_{(1)}, k_{(2)}] = ak_{(1)} + bk_{(2)} + ck_{(3)}, \tag{6.4}$$

$$[k_{(1)}, k_{(3)}] = dk_{(1)} + ek_{(2)} + fk_{(3)}, \tag{6.5}$$

$$[k_{(2)}, k_{(3)}] = gk_{(1)} + hk_{(2)} + jk_{(3)}, \tag{6.6}$$

where $a, \ldots, j$ are constants to be determined from (6.1)–(6.6).

Systematic investigation of solutions of the set (6.4)–(6.6) leads to 13 inequivalent cases (see [25] for details) containing all Bianchi types from I to VI$_h$. Among them are special cases of the solutions found by Ellis [4] (which are of Bianchi types I, II and III), the solutions of Bianchi type I by Maitra [3], King [5], Vishveshwara and Winicour [6] and Ozsváth [21], and three simple new explicit solutions of unknown interpretation (of Bianchi types II and III).

The solutions of Bianchi type I from this class will not contain any generalizations of the flat Friedmann model because the velocity field of the rotating dust is tangent to the symmetry orbits here, i.e. these space-times are stationary. However, there are two Bianchi type V classes here, one of which is known to contain the $k < 0$ Friedmann model as a subcase of the limit $\omega = 0$. This one will be briefly described in Section 8 below. The Bianchi types VI$_h$ and IV of this class have so far not been explored at all in the literature and their physical interpretation is unknown.

7. The case of all three generators being linearly independent of u^α and w^α

The Killing fields $k_{(1)}$ and $k_{(2)}$ are the same as in (6.1)–(6.2), while $k_{(3)}$ is

$$k^\alpha_{(3)} = (C_3 + \psi - y\psi_{,y})\delta^\alpha_0 + \psi_{,y}\delta^\alpha_1 - \psi_{,x}\delta^\alpha_2 + \lambda_3(x,y)\delta^\alpha_3; \tag{7.1}$$

ϕ and ψ are functions of x and y. The commutator equations (6.4)–(6.6) can be partly integrated without going into separate cases, resulting in the set

$$\phi_{,x} = ay + b\phi + c\psi + bC_2 + cC_3; \tag{7.2}$$

$$\psi_{,x} = dy + e\phi + f\psi + eC_2 + fC_3; \tag{7.3}$$

$$\phi_{,y}\psi_{,x} - \phi_{,x}\psi_{,y} = gy + h\phi + j\psi + C, \tag{7.4}$$

$$\lambda_{2,x} = b\lambda_2 + c\lambda_3, \qquad \lambda_{3,x} = e\lambda_2 + f\lambda_3, \tag{7.5}$$

$$\begin{aligned}\phi_{,y}(\psi_{,x} - y\psi_{,xy}) + y\phi_{,x}\psi_{,yy} - \psi_{,y}(\phi_{,x} - y\phi_{,xy}) - y\psi_{,x}\phi_{,yy} \\ = h(C_2 + \phi - y\phi_{,y}) + j(C_3 + \psi - y\psi_{,y}),\end{aligned} \tag{7.6}$$

where C is an arbitrary constant.

The equations to solve are (7.2), (7.3) and (7.5); the remaining ones are consistency conditions to be imposed on the solutions. The set (7.2)–(7.3) and the set (7.5) are both (ordinary differential) linear vector equations of the same form

$$U_{,x} = AU + W, \tag{7.7}$$

where, for (7.2)–(7.3), the constant matrix A and the vectors U and W are

$$A = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \qquad U = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \qquad W = y\begin{pmatrix} a \\ d \end{pmatrix} + A\begin{pmatrix} C_2 \\ C_3 \end{pmatrix}, \tag{7.8}$$

while for (7.5) the matrix A is the same, $W = 0$ and $U = \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}$.

Solving Eq. (7.8) is a textbook exercise, but, since the constants $a, \ldots, j$ are all arbitrary, a multitude of separate cases arises: the matrix A may be nonsingular with two complex eigenvalues, two real eigenvalues, one double eigenvalue or a single eigenvalue, it may be singular with two different eigenvalues, nilpotent, etc. Some of the subcases turn out to be equivalent in the end (in the sense that they generate the same algebra), some others turn out to be reducible (by changes of the basis of Killing vectors) to those considered in Section 6. In the end, the number of cases to be considered separately is 10, among them all Bianchi types except II, III and V [26]. The Bianchi type I case will not contain any generalization of the flat Friedmann model for the same reason as before (the velocity field is tangent to the symmetry orbits), but, nevertheless, there is room for all three types of Friedmann models here because the Bianchi types VII_0, VII_h and IX are all represented. However, they are all hopelessly complicated and so far nothing is known about them.

This whole family of cases has not been explored in literature up to now except for a special Bianchi type VII_0 model whose asymptotic properties were considered by Demiański and Grishchuk [28]. This special case has flat hypersurfaces $t = \text{const}$.

8. A promising Bianchi type V case

One of the two Bianchi type V metrics obtained in the class of Section 6 is the following

$$\begin{aligned} ds^2 = (dt + ydx)^2 - (yk_{11})^2 dx^2 - (k_{22}/y)^2(dy + y^2 h dx)^2 \\ - {k_{33}}^2[yf dx + (g/y)dy + ydz]^2, \end{aligned} \tag{8.1}$$

where $k_{11}(t), k_{22}(t), h(t), k_{33}(t), f(t)$ and $g(t)$ are functions of t to be found from the Einstein equations. (This is the Bianchi type V subcase of the case 1.2.2.2 of [25] simplified by a coordinate transformation.) It is most promising because some progress toward integrating the Einstein equations was made here (to be published), while the $k < 0$ Friedmann model can be seen to be a subcase of the limit of zero rotation. This will be shown below.

As is obvious from Eqs. (3.20) and (3.22), the Plebański coordinates are ill-suited for considering the limit $\omega = 0$. Hence, for calculating this limit, a coordinate transformation has to be carried out first. The limit subsequently calculated will be nonunique (i.e. another nonrotating limit might be obtained starting from a different coordinate transformation). However, we will be satisfied to show that a limit exists, in which the $k < 0$ Friedmann model is contained.

Let ω_0 be a constant and let

$$y = \omega_0 y'. \tag{8.2}$$

Then, in the new coordinates

$$\omega'_{12} = \omega_0 \omega_{12} \tag{8.3}$$

and $\omega_0 \to 0$ will result in $\omega_{\alpha\beta} \equiv 0$. However, this limit would make the metric tensor singular, and so it has to be accompanied by a few reparametrizations (this is where nonuniqueness enters again). The following reparametrizations will remove the problem

$$k_{11} = \tilde{k}_{11}/\omega_0, \qquad k_{33} = \tilde{k}_{33}/\omega_0, \qquad g = \tilde{g}\omega_0, \qquad n = \tilde{n}{\omega_0}^2 \tag{8.4}$$

(the last reparametrization assures that also the rotation vector $w^\alpha \to 0$ in the limit $\omega_0 \to 0$.)

After the reparametrization the metric becomes

$$\begin{aligned} ds^2 &= (dt + \omega_0 y' dx)^2 - (y'\tilde{k}_{11} dx)^2 - (k_{22}/y')^2 (dy' + \omega_0 y'^2 h dx)^2 \\ &\quad - \tilde{k}_{33}^2 [y' f dx + (\tilde{g}/y') dy' + y' dz]^2 \quad \xrightarrow[\omega_0 \to 0]{} \\ &dt^2 - (y'\tilde{k}_{11} dx)^2 - (k_{22}/y')^2 dy'^2 - \tilde{k}_{33}^2 [y' f dx + (\tilde{g}/y') dy' + y' dz]^2, \end{aligned} \tag{8.5}$$

and it is no longer singular and even more general than necessary. The $k < 0$ Friedmann model results from here as the following subcase

$$f = \tilde{g} = 0, \qquad \tilde{k}_{11} = \tilde{k}_{22} = \tilde{k}_{33} := R(t), \tag{8.6}$$

where $R(t)$ is the Friedmann scale factor. The standard forms of the Friedmann metric can be obtained from the above by elementary coordinate transformations.

It was shown above that the $k < 0$ Friedmann model emerges when the limit $\omega_0 \to 0$ is taken in the Einstein equations. This does not yet guarantee that *solutions* of Einstein's equations will have the same property. It happens that solutions of nonlinear differential equations do not form continuous families, but rather branched trees: alternatives appear while solving the equations, and after one branch of the alternative is chosen, the other branch is not accessible through limiting transitions. In fact, an example is seen in Section 5 of this text: the Bianchi type I solutions have symmetry generators that result from those for Bianchi type II in the limit $b \to 0$. However, the collection of solutions for Bianchi type I is larger than the $b \to 0$ limit of the type II solutions. The same situation may happen with the metric (8.1). Hence, an explicit solution of the Einstein equations has to be found for this metric and then its limit $\omega_0 \to 0$ has to be compared with the Friedmann model. Work on this is in progress.

9. Conclusion

This investigation should be useful as an intermediate step in looking for more general solutions: perfect fluid solutions with the same symmetries and any solutions with lower symmetries. The progress with respect to earlier knowledge on hypersurface-homogeneous geometries with a rotating dust source consists in the fact that such solutions have been looked for by trial and error, beginning from certain metric ansatzes. The collection of possible ansatzes was hereby reduced to a well-defined, not-too-large set.([1])

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([1]) The algebraic calculations for this paper were done with use of the program Ortocartan [22, 23].

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