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The Ultimate Extension of the Bianchi Classification for Rotating Dust Models

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ABSTRACT For a rotating dust with a 3-dimensional symmetry group all possible metric forms can be classified and, within each class, explicitly written out. With respect to the structures of the groups, this is just the Bianchi classification, but with all possible orientations of the orbits taken into account. This result follows from the formalism introduced by Plebański. This paper is a brief overview of results that will be published elsewhere.

20.1 Introduction and Summary

The theorem of Darboux (see Section 20.2) allows one to introduce invariantly defined coordinates in which the velocity field of a fluid acquires a preferred form. It is assumed in addition that the fluid moves with zero acceleration and nonzero rotation. These assumptions result in a simplification of the metric tensor and in limitations imposed on the Killing vectors, if any exist. A Killing field k^α may be spanned on velocity u^α and rotation w^α or may be linearly independent of u^α and w^α. This gives rise to a classification of possible symmetries in the presence of rotating matter.

When there exist three linearly independent Killing fields, the classification described above gives rise to a complete classification of all possible metric forms. With respect to the algebras of the symmetry groups, this is just the Bianchi classification, but with all possible orientations of the orbits taken into account (i.e., they may be timelike, spacelike or null).

In every case that emerges, the commutation relations of the algebra have been solved, resulting in explicit formulae for the Killing fields, and then the Killing equations have been solved, resulting in the formulae for the metric tensors compatible with the symmetry group considered. The degree of success in solving the Einstein equations varied very strongly from case to case. In most cases, no progress was made. In some cases the Einstein equations have been integrated either to an autonomous set of first-order equations or to a single nonlinear differential equation of second or third order. Several solutions known earlier were identified in the present scheme (those by Lanczos [1], Gödel [2], Maitra [3], Ellis [4], Vishveshwara and Winocour [6], and a few solutions with rotating charged dust; see below).

The Darboux theorem was first applied as a tool for investigating the equations of motion and the Einstein equations by Plebański [7]. The approach of Plebański was used by this author [8–12] to find a collection of stationary, cylindrically symmetric solutions of Einstein's equations with berotropic perfect fluid sources.

In Section 20.2 the Darboux theorem and the associated classification of first-order differential forms are introduced. In Section 20.3 the classification is applied to geodesic vector fields with rotation. When the vector field is the velocity field of a fluid, a class of preferred coordinates results (which shall be termed "Plebański coordinates"). In Section 20.4 it is shown that each Killing vector field that might possibly exist is a spacetime with a geodesic and rotating fluid source is determined by two functions of two variables. If the Killing field is not spanned on velocity and rotation, then the Plebański coordinates may be adapted to it so that it acquires the unique form $k^\alpha = \delta^\alpha_1$.

In Section 20.5, the consideration of Section 20.4 is applied to the case of three Killing vector fields existing on a manifold. When all three of them are spanned on u^α and w^α, the group becomes 2-dimensional, and this case is not considered here. When two of them are spanned on u^α and w^α while the third one is not, two cases arise. In one of them (Bianchi type II), the Einstein equations reduce to a single ordinary differential equation of third order (of second order when $\Lambda = 0$). In the other case (Bianchi type I), the Einstein equations are reduced to an autonomous set of first-order equations. The solutions of Lanczos [1] and of Gödel [2] emerge as special cases of both these classes.

The remaining cases are sketched only briefly. Section 20.6 contains the description of the case when two of the Killing fields are linearly independent of u^α and w^α; Section 20.7 of the case when all three Killing fields are linearly independent of u^α and w^α. With the increasing number of Killing vectors that are linearly independent of u^α and w^α, the equations become progressively more complicated, the number of subcases requiring a separate treatment increases, while the progress in integrating the Einstein equations decreases. The full results of the investigation will be published elsewhere.

20.2 The Classification of Differential Forms of First Order and the Darboux Theorem

Definition. *Let q be a differential form of first order.*

If $Q_{2l} := dq \wedge \ldots \wedge dq$ [multiplied l times] $\neq 0$, but $q \wedge Q_{2l} = 0$, then q is said to be of class $2l$.

If $Q_{2l+1} := q \wedge Q_{2l} \neq 0$, but $dQ_{2l+1} \equiv dq \wedge Q_{2l} = 0$, then q is said to be of class $(2l+1)$. □

Then the following holds:

The Theorem of Darboux. *The form q is of class $2l$ if and only if there exists a set of $2l$ independent functions $(\xi_1, \dots, \xi_l, \eta_1, \dots, \eta_l)$ such that*

$$q = \eta_1 d\xi_1 + \eta_2 d\xi_2 + \dots + \eta_l d\xi_l. \tag{1}$$

The form q is of class $(2l+1)$ if and only if there exists a set of $(2l+1)$ independent functions $(\tau, \xi_1, \dots, \xi_l, \eta_1, \dots, \eta_l)$ such that

$$q = d\tau + \eta_1 d\xi_1 + \eta_2 d\xi_2 + \dots + \eta_l d\xi_l. \tag{2}$$

□

(See [13] for a proof).

The Darboux theorem implies that in a 4-dimensional spacetime V_4 the most general differential form of first order can be represented as

$$q = \sigma d\tau + \eta\, d\xi, \tag{3}$$

where σ, τ, η, and ξ are scalar functions on V_4.

Any vector field u^α on V_4 defines the following form of first order:

$$q_u := u_\alpha\, dx^\alpha. \tag{4}$$

According to (3), in the most general case there exist scalar functions σ, τ, η, and ξ such that

$$u_\alpha = \sigma\tau,_\alpha + \eta\xi,_\alpha. \tag{5}$$

In general, the four functions are independent, i.e.,

$$\frac{\partial(\sigma, \tau, \eta, \xi)}{\partial(x^0, x^1, x^2, x^3)} \neq 0. \tag{6}$$

In that case, they can be chosen as coordinates in the spacetime.

20.3 Geodesically Moving Fluids

To any timelike vector field u_α normalized to unity (so that $u_\alpha u^\alpha = 1$), the decomposition described in [14] and [15] may be applied:

$$u_{\alpha;\beta} = \dot{u}_\alpha u_\beta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}, \tag{7}$$

where $\dot{u}^\alpha$, θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are, respectively, acceleration, expansion, shear, and rotation. In the signature (+ - - -) used here, the projection tensor $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta. \tag{8}$$

The following equations hold:

$$\dot{u}_\alpha u^\alpha = 0, \qquad \sigma_{\alpha\beta} u^\beta = \omega_{\alpha\beta} u^\beta = 0. \tag{9}$$

We shall assume from now on that u_α is the velocity field of a fluid and that $\dot{u}_\alpha = 0$, i.e., that the particles of the fluid move on geodesics. Then, from (5) we have

$$\omega_{\alpha\beta} = \frac{1}{2}(\sigma,_\beta \tau,_\alpha - \sigma,_\alpha \tau,_\beta + \eta,_\beta \xi,_\alpha - \eta,_\alpha \xi,_\beta), \tag{10}$$

and from (9) we have

$$(u^\beta \sigma,_\beta)\tau,_\alpha - (u^\beta \tau,_\beta)\sigma,_\alpha + (u^\beta \eta,_\beta)\xi,_\alpha - (u^\beta \xi,_\beta)\eta,_\alpha = 0. \tag{11}$$

It is easy to see that, in virtue of (11), the form (4) cannot be of class 4. Hence, for a geodesically moving fluid the form (4) is of class at most 3, i.e., at most 3 independent functions τ, η, ξ exist such that

$$u_\alpha = \tau,_\alpha + \eta\xi,_\alpha . \tag{12}$$

The functions $\{\tau, \xi, \eta\}$ in (12) are determined up to the following transformations:

$$\xi = F(\xi', \eta'), \qquad \eta = G(\xi', \eta'), \tag{13}$$

$$\tau = \tau' - S(\xi', \eta'), \tag{14}$$

where the functions F and G must obey the equation

$$F,_{\xi'} G,_{\eta'} - F,_{\eta'} G,_{\xi'} = 1, \tag{15}$$

and then S is determined by

$$S,_{\xi'} = GF,_{\xi'} - \eta', \qquad S,_{\eta'} = GF,_{\eta'} . \tag{16}$$

Let us now make the additional assumption that the number of particles of the fluid is conserved, i.e.,

$$(\sqrt{-g}nu^\alpha)_{,\alpha} = 0, \tag{17}$$

where g is the determinant of the metric tensor and n is the particle number density. This equation is a necessary and sufficient condition for the existence of a function ζ such that

$$\sqrt{-g}nu^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\xi,_\beta \eta,_\gamma \zeta,_\delta . \tag{18}$$

Note that (12) implies that

$$u^\alpha \tau,_\alpha = 1, \tag{19}$$

and then Eq. (18) implies that

$$\varepsilon^{\alpha\beta\gamma\delta}\tau,_\alpha \xi,_\beta \eta,_\gamma \zeta,_\delta \equiv \frac{\partial(\tau, \eta, \xi, \zeta)}{\partial(x^0, x^1, x^2, x^3)} = \sqrt{-g}n \neq 0. \tag{20}$$

Equation (18) also implies that

$$u^\alpha \zeta,_\alpha = 0. \tag{21}$$

The function ζ is determined by (18) up to the transformations

$$\zeta = \zeta' + T(\xi', \eta'), \tag{22}$$

where T is an arbitrary function. Equation (20) certifies that $\{\tau, \xi, \eta, \zeta\}$ can be used as coordinates in the spacetime. If they are chosen as the $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ coordinates, respectively, then Eq. (12) implies

$$u_0 = 1, \qquad u_1 = y, \qquad u_2 = u_3 = 0. \tag{23}$$

We will use these coordinates throughout the remaining part of the paper and call them "Plebański coordinates." Equation (20) implies now

$$g = -n^{-2}, \tag{24}$$

and Eq. (18) implies

$$u^\alpha = \delta^\alpha_0, \tag{25}$$

i.e., the Plebański coordinates are comoving. The rotation vector defined by

$$w^\alpha = -(1/\sqrt{-g})\varepsilon^{\alpha\beta\gamma\delta} u_\beta u_{\gamma,\delta} \tag{26}$$

assumes the form

$$w^\alpha = n\delta^\alpha_3. \tag{27}$$

Equations (23) and (25) imply that

$$g_{00} = 1, \qquad g_{01} = y, \qquad g_{02} = g_{03} = 0, \tag{28}$$

and also that the only nonvanishing components of the rotation tensor are

$$\omega_{12} = -\omega_{21} = 1/2. \tag{29}$$

If we now assume that the fluid is perfect, then we conclude from the equations of motion $T^{\alpha\beta}_{;\beta} = 0$ that either $\omega = 0$ or $p = \text{const.}$ (see also [16]). This means that a geodesic perfect fluid can be rotating only if it is in fact dust; the constant p can be reinterpreted as the cosmological constant. In this case, the energy-density obeys the conservation equation $(\sqrt{-g}\epsilon u^\alpha)_{,\alpha} = 0$ and Eq. (17) need not be assumed separately. A more detailed exposition of the same material can be found in [8].

Note that the rotating dust is not the only example to which this approach may be applied. In several papers, rotating charged dust was considered under the additional assumptions that all charges are attached to the dust particles, that the only current is the one created by the flow of dust, and that the Lorentz force acting on the dust particles $F^\mu{}_\nu u^\nu$ is zero, (i.e., that the electric and magnetic fields are such that they cancel each other's influence on the charged dust particles). Under these assumptions the dust particles move on geodesics and the formalism of this section applies. Such solutions of the Einstein-Maxwell equations were considered in references [18–25]; they will be mentioned in Section 20.5.

20.4 The Killing Vector Fields Compatible with Rotation

We shall assume that the symmetries of the spacetime (if any exist) are inherited by the source, i.e., that if the Lie derivative of the metric tensor $g_{\alpha\beta}$ along the vector field k^α is zero, $\pounds_k g_{\alpha\beta} = 0$, then the velocity field and the particle number density are also invariant: $\pounds_k u^\alpha = 0 = \pounds_k n$. (For a pure perfect fluid source the inheritance is guaranteed.) It follows that the rotation tensor must also be invariant, $\pounds_k \omega_{\alpha\beta} = 0$. All these equations imply that

$$k^0 = C + \phi - y\phi,_y\,, \qquad k^1 = \phi,_y\,, \qquad k^2 = -\phi,_x\,, \qquad k^3 = \lambda, \tag{30}$$

where $\phi(x, y)$ and $\lambda(x, y)$ are arbitrary functions and C is an arbitrary constant. If there are no symmetries, then $\phi = \lambda = C = 0$. However, if any symmetries are present, then the Killing vector fields must have the form (30).

Suppose that ϕ is not a constant, i.e., that a Killing vector field k^α exists that has a nonzero component in the x- or y-direction (in invariant terms this means that the vector field k^α is not spanned on the vector fields of velocity, u^α, and rotation, w^α). We can then, within the Plebański class defined in Section 20.3, adapt the coordinates to k^α in such a way that $k^{\alpha'} = \delta^{\alpha'}_1$, i.e., so that the metric becomes independent of x'. From (13)–(16) and (22) the transformation functions are

$$t' = t - S(x, y), \qquad x' = F(x, y), \qquad y' = G(x, y), \qquad z' = z + T(x, y), \tag{31}$$

where T is arbitrary, while F, G, and S obey

$$F,_x G,_y - F,_y G,_x = 1, \qquad S,_x = GF,_x - y, \qquad S,_y = GF,_y\,. \tag{32}$$

The condition $k^{\alpha'} = \delta^{\alpha'}_1$ then implies

$$G = \phi + C, \tag{33}$$

$$T,_x \phi,_y - T,_y \phi,_x = -\lambda. \tag{34}$$

Equations (32) and (34) simply define the accompanying F, S, and T, which are seen to exist always. Since ϕ was assumed nonconstant, the transformation is nonsingular, and results in $\phi = y$ in the new coordinates; the metric becomes independent of x after the transformation. This property is preserved by the transformations (31), but with F, G, S, and T restricted now by

$$G = y, \qquad F = x + H(y), \qquad T = T(y), \qquad S = \int yH,_y\, dy + A, \tag{35}$$

where A is an arbitrary constant and H and T are arbitrary functions.

Suppose that three Killing vector fields exist and all three are spanned on u^α and w^α, so that ϕ = const. in (30) for each of them, i.e.,

$$k^\alpha_{(i)} = C_i \delta^\alpha_0 + \lambda_i(x, y)\delta^\alpha_3, \qquad i = 1, 2, 3. \tag{36}$$

From the Killing equations it follows then that constants α_1, α_2 and α_3 exist such that $\alpha_1 k_{(1)} + \alpha_2 k_{(2)} + \alpha_3 k_{(3)} = 0$, i.e., the symmetry group is in fact 2-dimensional. Hence, for a 3-dimensional group at least one of the generators must be linearly independent of u^α and w^α at every point of the spacetime region under consideration.

20.5 The Case of Two Generators Spanned on u^α and w^α

In this section we shall assume that exactly one generator, $k^\alpha_{(1)}$, is linearly independent of u^α and w^α, while the other two, $k^\alpha_{(2)}$ and $k^\alpha_{(3)}$, are of the form (36). In agreement with the result of Section 20.4, the Plebański coordinates can be adapted to $k^\alpha_{(1)}$ so that

$$k^\alpha_{(1)} = \delta^\alpha_1, \tag{37}$$

while

$$k^\alpha_{(2)} = C_2 \delta^\alpha_0 + \lambda_2(x, y)\delta^\alpha_3, \qquad k^\alpha_{(3)} = C_3 \delta^\alpha_0 + \lambda_3(x, y)\delta^\alpha_3, \tag{38}$$

and the coordinate transformations preserving (37) and (38) are (35). Note that C_2 and C_3 cannot vanish simultaneously because otherwise the Killing equations immediately imply that either $k^\alpha_{(3)} = \text{const.}\ k^\alpha_{(2)}$ (in which case the symmetry group is 2-dimensional) or the metric is singular. However, with no loss of generality we can assume that

$$C_2 \neq 0 = C_3 \tag{39}$$

because the Killing vector fields are determined up to linear combinations among them. This implies $\lambda_3 \neq 0$. The commutators of the Killing vectors are then

$$[k_{(1)}, k_{(2)}]^\alpha = (\lambda_{2,x}/\lambda_3)k^\alpha_{(3)}, \qquad [k_{(2)}, k_{(3)}]^\alpha = 0, \tag{40}$$

$$[k_{(1)}, k_{(3)}]^\alpha = (\lambda_{3,x}/\lambda_3)k^\alpha_{(3)}. \tag{41}$$

The Killing vector fields will thus form a Lie algebra when

$$\lambda_{2,x} = b\lambda_3, \qquad \lambda_{3,x} = c\lambda_3, \tag{42}$$

where b and c are arbitrary constants. The cases $c \neq 0$ and $c = 0$ have to be considered separately. However, when $c \neq 0$, it follows that

$$\lambda_3 = \beta(y)e^{cx}, \qquad \lambda_2 = (b/c)\beta(y)e^{cx} + \alpha(y), \tag{43}$$

and so with no loss of generality we can assume $b = 0$. The Einstein equations then imply that either $c = 0$ or there is no rotation. Since we are interested in rotating solutions only, this case need not be followed. We thus consider

Case I: $c = 0, b \neq 0$. Then

$$\lambda_3 = \beta(y), \qquad \lambda_2 = b\beta(y)x + \alpha(y). \tag{44}$$

The algebra of the Killing vector fields is of Bianchi type II when $b \neq 0$ and of Bianchi type I when $b = 0$.

In order to simplify the Killing vectors, we now transform the coordinates as follows:

$$(t', x', y') = (t, x, y), \qquad z' = -(\alpha/C_2)t + z/\beta. \tag{45}$$

The transformation is not of the form (35), so the new coordinates do not belong to the Plebański class, and the forms of velocity, rotation, and the metric will no longer agree with (23) - (29). The Killing vector fields in the new coordinates become

$$k^{\alpha}_{(1)} = \delta^{\alpha}_{1}, \qquad k^{\alpha}_{(2)} = \delta^{\alpha}_{0} + bx\delta^{\alpha}_{3}, \qquad k^{\alpha}_{(3)} = \delta^{\alpha}_{3}, \tag{46}$$

while the velocity and rotation fields become

$$u^{\alpha} = \delta^{\alpha}_{0} - (\alpha/C_2)\delta^{\alpha}_{3}, \qquad w^{\alpha} = (n/\beta)\delta^{\alpha}_{3}. \tag{47}$$

The transformed metric is independent of x and z.

The orbits of the symmetry group are the $\{y = \text{const}\}$ hypersurfaces. In order to follow the standard technique of the Bianchi-type spaces, we should now carry out a coordinate transformation that preserves (46) and makes the y-coordinate curves orthogonal to the group orbits, so that $g'_{02} = g'_{12} = g'_{23} = 0$ after the transformation. This step is not in fact necessary for solving the Einstein equations (in general it only reshuffles the unknown functions without eliminating any of them), but in the case under consideration it leads to a simplification.

After the transformation, and with the Killing equations solved, the metric becomes (primes dropped; details of the derivation will be given in another paper):

$$ds^2 = (dt + Y\,dx)^2 - (F\,dx)^2 - dy^2 - G^2[A\,dt - (bt - k_{13})\,dx + dz]^2, \tag{48}$$

where $G(y)$, $A(y)$, $k_{13}(y)$, and $F(y)$ are new names for the unknown functions. The velocity field in the coordinates of (48) is

$$u^{\alpha} = \delta^{\alpha}_{0} - A\delta^{\alpha}_{3}. \tag{49}$$

The components of the Einstein tensor will be referred to the orthonormal tetrad of forms $e^i = e^i_{\alpha}\,dx^{\alpha}$, $i = 0, 1, 2, 3$, uniquely implied by (48). Note that $e^0 = u_{\alpha}\,dx^{\alpha}$.

The equation $G_{12} = 0$ implies that $bA,_y = 0$. The case $b = 0$ will be considered separately below, so we take here

$$A = \text{const}. \tag{50}$$

Then other field equations, together with simplifying coordinate transformations, lead to

$$A = k_{13} = 0, \tag{51}$$

$$Y,_y\,G/F = B = \text{const}. \tag{52}$$

and we can assume $B \neq 0$ because rotation would be zero with $B = 0 = Y,_y$.

It is convenient to introduce $Y(y)$ as the new variable. The equation $G_{11}+G_{22} = 2\Lambda$ can then be written, with the help of (52), as

$$(F^2 G,_Y /G),_Y = \frac{2\Lambda G^2}{B^2} - \frac{1}{2}, \tag{53}$$

and so

$$F^2 = \left(C - \frac{1}{2}Y + 2\frac{\Lambda}{B^2}\int G^2\, dY\right) G/G,_Y\,, \tag{54}$$

where C is a new arbitrary constant (we can assume $G,_Y \neq 0$ because $G,_Y = 0$ immediately implies $b = 0$ from $G_{11} - G_{22} = 0$, and $b = 0$ will be considered separately). Using (54) in $G_{22} = \Lambda$, we obtain the following integro-differential equation that determines G:

$$-\frac{1}{4}b^2 GG,_Y + \frac{1}{2}(B/G)^2\left(C - \frac{1}{2}Y + 2\frac{\Lambda}{B^2}\int G^2\, dY\right)^2 (G,_Y /G - G,_{YY} /G,_Y) = 0. \tag{55}$$

In the special case $\Lambda = 0$, this becomes an ordinary second-order differential equation. It is easy to get rid of the integral by transforming (55) appropriately and differentiating the result by Y (in this way a third-order differential equation for $G(Y)$ is obtained) or by introducing the new variable $u(Y)$ by $dY/du = 1/G^2$ (this results in a second-order equation for $G(u)$). However, no progress toward solving (55) results in either case.

The formula for energy-density may be simplified to

$$(8\pi G/c^4)\epsilon = (B/G)^2 - (bG)^2 - 2\Lambda. \tag{56}$$

Note that the solutions considered here have a meaningful limit $b = 0$.

When $G =$ const., equations (54) and (55) no longer apply and one has to go back to the Einstein equations; the resulting metric is the Gödel solution.

When $G,_Y \neq 0 = b$, Eq. (55) implies $G = e^{DY+E}$, and this leads to the Lanczos solution (see Ref. [8]).

Case II: $b = c = 0$ in (40)–(42).

The reasoning up to Eq. (49) applies also here, but (50) no longer follows. Instead, the equation $G_{13} = 0$ can be integrated with the result

$$k_{13,y} = BF/G^3 - YA,_y\,, \tag{57}$$

where B is an arbitrary constant; the equation $G_{01} =$ can be integrated to

$$Y,_y = (C - BA)F/G, \tag{58}$$

where C is an arbitrary constant; and the equation $G_{03} = 0$ can be integrated to

$$A,_y = (BY - D)/(FG^3), \tag{59}$$

where D is one more arbitrary constant.

At this point, only the diagonal components of the Einstein tensor survive, and $G_{00} = (8\pi G/c^4)\epsilon - \Lambda$ just defines the energy-density. The equations $G_{11} = \Lambda =$

$G_{22} = G_{33}$ can be integrated to the first-order set[1]

$$FG,_y = \frac{1}{2}Bk_{13} + \frac{1}{2}BAY - \frac{1}{2}CY + 2\Lambda \int FG\,dy + H_0, \tag{60}$$

$$GF,_y = -\frac{1}{2}Bk_{13} - \frac{1}{2}DA - E + 2\Lambda \int FG\,dy + H_0. \tag{61}$$

where H_0 is an arbitrary constant. The integral can be calculated if the new variable $u(y)$ is introduced by

$$dy/du = 1/(FG). \tag{62}$$

In terms of the variable u from (61), equations (57)–(61) form an autonomous set of first-order equations that can be investigated further by qualitative methods (see for example [17]). This is left as a subject for a separate study.

The functions $A(y)$ and $k_{13}(y)$ have invariant meaning: they are proportional to the scalar products of the Killing vectors (see equations (46) and (48) with $b = 0$):

$$A = -g_{\alpha\beta}k^{\alpha}_{(2)}k^{\beta}_{(3)}/G^2, \qquad k_{13} = -g_{\alpha\beta}k^{\alpha}_{(1)}k^{\beta}_{(3)}/G^2 \tag{63}$$

(note that $G^2 = -g_{\alpha\beta}k^{\alpha}_{(3)}k^{\beta}_{(3)}$, i.e., it is a scalar, too). Hence, $A = 0$ and $k_{13} = 0$ are invariant properties. Note that $A = 0$ implies, through (59), that either $Y =$ const. (in which case there is no rotation) or $B = D = 0$. In the latter case, $k_{13} =$ const. and the coordinate transformation $z = z' - k_{13}x$ leads to $k_{13} = 0$ in the new coordinates. With $A = k_{13} = 0$, the Lanczos and Gödel models result from the Einstein equations as the only solutions.

In references [18–25], charged dust solutions with zero Lorentz force were considered. Apart from the one in [18], they are cylindrically symmetric and stationary, and so they would emerge in this section, had we allowed charged dust as a source and considered the Einstein-Maxwell equations. However, not all of these solutions allow nonempty limits of vanishing electromagnetic fields. The ones from [19] and [21] become a vacuum solution and the Minkowski spacetime, respectively, in that limit, the one from [20] does not allow the limit at all. The solution by Som and Raychaudhuri [22] is a generalization to charged dust of the $\Lambda = 0$ subcase of the Lanczos solution [1]; the first of the six solutions by Banerjee and Banerji [23] is a generalization of the Gödel solution. (The other solutions from Ref. [23] have the following properties: the second and the sixth become vacuum solutions in the limit $F_{\mu\nu} = 0$, the third does not allow this limit at all, the fourth has a 2-dimensional symmetry group, and the fifth reduces to the Minkowski metric in the limit.) The two solutions by Mitskiévič and Tsalakou [24] are, respectively, generalizations of the full Lanczos solution and of the Gödel solution to a charged dust source, and those by Upornikov [25] are coordinate transforms of those from [24].

This short overview of literature deliberately omits plain rediscoveries of solutions known earlier; they will be listed in the main paper.

[1] The derivation will be published separately.

20.6 The Case of One Generator Spanned on u^α and w^α

The number of cases that require separate treatment is larger here, and the equations are more complicated, so the results will be described only briefly.

One of the two generators that are not spanned on u^α and w^α may be given the simple form (37) by a coordinate transformation, the other will have the general form (30). The one that is spanned on u^α and w^α will have the form (38). Hence, the generators are

$$k^\alpha_{(1)} = \delta^\alpha_1,$$

$$k^\alpha_{(2)} = (C_2 + \phi - y\phi,_y)\delta^\alpha_0 + \phi,_y \delta^\alpha_1 - \phi,_x \delta^\alpha_2 + \lambda_2(x, y)\delta^\alpha_3,$$

$$k^\alpha_{(3)} = C_3\delta^\alpha_0 + \lambda_3(x, y)\delta^\alpha_3, \tag{64}$$

and the remaining freedom of coordinate transformations is given by equations (31) and (35). The Killing fields (64) will form a Lie algebra when

$$[k_{(1)}, k_{(2)}] = ak_{(1)} + bk_{(2)} + ck_{(3)},$$

$$[k_{(1)}, k_{(3)}] = dk_{(1)} + ek_{(2)} + fk_{(3)},$$

$$[k_{(2)}, k_{(3)}] = gk_{(1)} + hk_{(2)} + jk_{(3)}, \tag{65}$$

where $a, \ldots, j$ are constants to be determined from (64) and (65). The set (65) written out explicitly is

$$\begin{aligned}
&\phi,_x - y\phi,_{xy} = b(C_2 + \phi - y\phi,_y) + cC_3,\\
&\phi,_{xy} = a + b\phi,_y, \qquad \phi,_{xx} = b\phi,_x,\\
&\lambda_{2,x} = b\lambda_2 + c\lambda_3,\\
&e(C_2 + \phi - y\phi,_y) + fC_3 = 0,\\
&d + e\phi,_y = 0, \qquad e\phi,_x = 0,\\
&\lambda_{3,x} = e\lambda_2 + f\lambda_3,\\
&h(C_2 + \phi - y\phi,_y) + jC_3 = 0,\\
&g + h\phi,_y = 0, \qquad h\phi,_x = 0,\\
&\phi,_y \lambda_{3,x} - \phi,_x \lambda_{3,y} = h\lambda_2 + j\lambda_3.
\end{aligned} \tag{66}$$

Two equations in this set form an alternative: either $\phi,_x = 0$ or $e = h = 0$. Such alternatives also occur at later stages of integration, and they give rise to the large number of separate cases. In three of them simple explicit solutions, most probably new, were derived; they will be published in the main papers. The case that contains the solutions considered by Maitra [3], King [5], and Vishveshwara and Winicour [6] will be presented here.

$$\phi,_x = \lambda_{2,x} = \lambda_{3,x} = C_3 = 0 \neq \lambda_3, \qquad \text{all constants } a, \ldots, j = 0. \tag{67}$$

Now $\phi,_y \neq 0$ follows from the definition of the case; otherwise $k_{(2)}$ would be spanned on u and w. The resulting Killing fields are given by the appropriate subcase of (64)–(65) and all commute to zero.[2] Hence, the Bianchi type of the algebra is I, but the solution is different from (57)–(61) because of the orientation of the orbit of the symmetry group in the spacetime.

Through coordinate transformations the Killing vector fields can now be transformed to the simplest form:

$$k^\alpha_{(1)} = \delta^\alpha_1, \qquad k^\alpha_{(2)} = \delta^\alpha_0, \qquad k^\alpha_{(3)} = \delta^\alpha_3, \tag{68}$$

and the velocity, rotation, and the metric then acquire the forms

$$\begin{aligned} u^\alpha &= F^{-1}(\delta^\alpha_0 - P\delta^\alpha_1 - \lambda_2\delta^\alpha_3), \qquad w^\alpha = (n/\lambda_3)\delta^\alpha_3, \\ ds^2 &= [(C_2 + \phi)dt + Y\mathrm{d}x]^2 - k^2_{11}(P\mathrm{d}t + dx)^2 - dy^2 \\ &\quad - k^2_{33}[(\lambda_2 + h_{13}P)dt + h_{13}dx + dz]^2, \end{aligned} \tag{69}$$

where all symbols are functions of y only, and

$$P = \phi,_Y \equiv \{\phi,_y\}/Y,_y, \qquad F = C_2 + \phi - YP. \tag{70}$$

King [5] considered rotating dust metrics with the same symmetry, but in addition assumed reflection symmetries that in the coordinates of (69) correspond to $z \to -z$ and $(t, x) \to (-t, -x)$. With these additional assumptions, $\lambda_2 = h_{13} = 0$. Even in this case, King found that the problem is underdetermined: one of the functions (in our notation it is k_{33}) may be chosen arbitrarily. King's paper contains a few examples of explicit solutions resulting from different choices of it (among them are the solutions of Lanczos [1] that goes by the name of Ehlers - van Stockum, and of Maitra [3]). Another specific example was found by Vishveshwara and Winicour [6].

King's metric ansatz can be derived from the following assumptions:

1. The algebra of the symmetry group is of Bianchi type I.
2. One Killing field ($k_{(3)}$) is collinear with rotation, the two others are linearly independent of u and w.
3. The velocity vector field is spanned on $k_{(1)}$ and $k_{(2)}$ only (i.e., $\lambda_2 = 0$).
4. The Killing fields $k_{(1)}$ and $k_{(2)}$ are both orthogonal to $k_{(3)}$ (i.e., $h_{13} = 0$).

Also in this class is the Maitra solution [3] which has the following invariant property in addition:

5. The timelike Killing field $k_{(2)}$ has unit length so that $(C_2 + \phi)^2 - k^2_{11}P^2 = 1$.

These conditions are still insufficient to reduce (68)–(69) to the Maitra solution; the following coordinate-dependent relations must hold in addition:

$$\begin{aligned} (C_2 + \phi)Y - k^2_{11}P &= m, \\ Y^2 + [1 - (C_2 + \phi)^2]/P^2 &= m^2 - r^2, \end{aligned} \tag{71}$$

[2] Some of the equations in (6.7) follow as necessary consequences of some others through (6.6), so the total number of different cases is not as large as (6.7) might suggest.

where $r(y)$ is a new coordinate defined by

$$\ln\left(\frac{dy}{dr}\right) = -\frac{1}{4u^2}\left\{(1+u^2)^{1/2} - 1 + \frac{1}{8} - \frac{1}{4}\ln[\frac{1}{2}(1+u^2)^{1/2} + \frac{1}{2}]\right\}$$

$$u := 2r/a, \qquad a = \text{const}, \tag{72}$$

and $m(r)$ is the function

$$m = -\frac{1}{2}a\left\{(1+u^2)^{1/2} - 1 - \ln\left[\frac{1}{2}(1+u^2)^{1/2} + \frac{1}{2}\right]\right\}. \tag{73}$$

This author was not able to interpret (71) in invariant terms.

The collection of models described in this section has a nonempty (in fact, quite large) common subset with those by Ellis [4]. However, the interrelations are somewhat complicated and require a more elaborate explanation. In short, the following classes of Ellis with nonzero rotation are not contained in this collection:

1. The generic case Ib; because it has a 4-dimensional symmetry group acting multiply transitively on 3-dimensional orbits, and the group has in general no 3-dimensional simply transitive subgroups. However, in special cases such subgroups do exist and the corresponding solutions of the Einstein equations are found in the present scheme.

2. The generic case Cii of the shearfree solutions; because it has only a 2-dimensional symmetry group.

All other rotating solutions of Ellis do belong to the present collection.

The first three of the six solutions by Ozsváth [26] also belong here, and they are subcases of the class considered by King. All of Ozsváth's solutions have 4-dimensional symmetry groups whose orbits are the whole 4-dimensional manifolds. In order to place specific Ozsváth's solutions in the classification considered here, one has to identify 3-dimensional subgroups of Ozsváth's groups. Examples can be spotted by inspection in Ref. 26 in which different non-isomorphic 3-dimensional subgroups are contained in the same 4-dimensional group. Hence, the same Ozsváth's solutions should come up as limits in different classes of the present investigation. For unique and complete identification, the formulae for group generators are necessary, and these are not given for most of Ozsváth's solutions. Among the cases that could not be identified are the other solutions of Ozsváth [27] and the "finite rotating Universe" of Ozsváth and Schücking [28-30]; this is where the present investigation makes contact with the legacy of the patron of this volume.

Other results in this class will be presented in a separate paper. Most of them do not lead to any explicit solution of the Einstein equations.

20.7 The Case of All Three Generators Being Linearly Independent of u^α and w^α

The commutator equations are still more complicated here, and the number of cases is still larger than in Section 20.6. The Killing fields $k_{(1)}$ and $k_{(2)}$ are the same as in (64), while $k_{(3)}$ is

$$k^\alpha_{(3)} = (C_3 + \psi - y\psi,_y)\delta^\alpha_0 + \psi,_y \delta^\alpha_1 - \psi,_x \delta^\alpha_2 + \lambda_3(x, y)\delta^\alpha_3; \tag{74}$$

ϕ and ψ are functions of x and y. The commutator equations (65) can be partly integrated without going into separate cases. The components (t, x, y) of the first equation in (65) are integrated to

$$\phi,_x = ay + b\phi + c\psi + bC_2 + cC_3; \tag{75}$$

the components (t, x, y) of the second equation in (65) are integrated to

$$\psi,_x = dy + e\phi + f\psi + eC_2 + fC_3; \tag{76}$$

and the components (x, y) of the third equation in (65) are integrated to

$$\phi,_y \psi,_x - \phi,_y \psi,_x = gy + h\phi + j\psi + C, \tag{77}$$

where C is an arbitrary constant. The remaining equations are

$$\lambda_{2,x} = b\lambda_2 + c\lambda_3, \qquad \lambda_{3,x} = e\lambda_2 + f\lambda_3, \tag{78}$$

$$\phi,_y (\psi,_x - y\psi,_{xy}) + y\phi,_x \psi,_{yy} - \psi,_y (\phi,_x - y\phi,_{xy}) - y\psi,_x \phi,_{yy}$$

$$= h(C_2 + \phi - y\phi,_y) + j(C_3 + \psi - y\psi,_y),$$

$$\phi,_y\lambda_{3,x} - \phi,_x\lambda_{3,y} - \psi,_y\lambda_{2,x} + \psi,_x\lambda_{2,y} = h\lambda_2 + j\lambda_3. \tag{79}$$

The equations to solve are (75), (76), and (78); the remaining ones are consistency conditions to be imposed on the solutions. The set (75)–(76) and the set (78) are both (ordinary differential) linear vector equations of the same form:

$$U,_x = AU + W, \tag{80}$$

where, for (75)–(76), the constant matrix A and the vectors U and W are

$$A = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad U = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad W = y\begin{pmatrix} a \\ d \end{pmatrix} + A\begin{pmatrix} C_2 \\ C_3 \end{pmatrix}, \tag{81}$$

while for (78) the matrix A is the same, $W = 0$, and $U = \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}$.

Solving Eq. (80) is a textbook exercise, but, since the constants $a, \ldots, j$ are all arbitrary, a multitude of separate cases arises: the matrix A may be nonsingular with two complex eigenvalues, with two real eigenvalues, one double eigenvalue or a single eigenvalue, it may be singular with two different eigenvalues, nilpotent, etc. Some of the subcases turn out to be equivalent in the end (in the sense that they generate the same algebra), some others turn out to be reducible (by changes of the basis of Killing vectors) to those considered in Section 20.6. Still, the number of cases to be considered is large. No explicit solutions of the Einstein equations were identified in this case, and so it will not be described here in any more detail.

20.8 Conclusion

The investigation should be useful as an intermediate step in looking for more general solutions: perfect fluid solutions with the same symmetries and any solutions with lower symmetries. The progress with respect to earlier knowledge on hypersurface-homogeneous geometries with a rotating dust source consists in the fact that such solutions have been looked for by trial and error, beginning from certain metric ansatzes. The collection of possible ansatzes was hereby reduced to a well-defined, not-too-large set.

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The algebraic calculation for this paper were done with use of the program Ortocartan [31, 32].
Note added in proof: The full results of the research reported here have already been published in *J. Math. Phys.* **39** (1998), 380–400, 401–422, 2148–2179.

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