Rotating dust solutions of Einstein's equations with 3-dimensional symmetry groups. I. Two Killing fields spanned on u^α and w^α

Andrzej Krasiński
N. Copernicus Astronomical Center and College of Science, Polish Academy of Sciences, Bartycka 18, 00 716 Warszawa, Poland

(Received 14 May 1997; accepted for publication 7 July 1997)

For a rotating dust with a 3-dimensional symmetry group all possible metric forms can be classified and, within each class, explicitly written out. This is made possible by the formalism of Plebański based on the Darboux theorem. In the resulting coordinates, the Killing vector fields (if any exist) assume a special form. Each Killing vector field may be either spanned on the fields of velocity and rotation or linearly independent of them. By considering all such cases one arrives at the classification. With respect to the structures of the groups, this is just the Bianchi classification, but with all possible orientations of the orbits taken into account. In this paper, which is part 1 of a 3-part series, all solutions are considered for which two Killing fields are spanned on velocity and rotation. The solutions of Lanczos and Gödel are identified as special cases, and their new invariant definitions are provided. In addition, a new invariant definition is given of the Ozsvath class III solution. © *1998 American Institute of Physics.* [S0022-2488(97)03112-5]

I. INTRODUCTION AND SUMMARY

The theorem of Darboux presented in Sec. II allows one to introduce invariantly defined coordinates in which the velocity field of a fluid (not assumed to be perfect) acquires a ''canonical'' form. In this paper it is further assumed that the fluid moves with zero acceleration and nonzero rotation. These assumptions result in a simplification of the metric tensor and in limitations imposed on the Killing vectors, if any exist. Within this special class of coordinates, any single Killing field may also be reduced to a ''canonical'' form, a different one in the case when it is spanned on the vector fields of velocity u^α and rotation w^α, and a different one when it is linearly independent of u^α and w^α. This gives rise to a classification of possible symmetries in rotating matter.

When there exist three linearly independent Killing fields, the classification described above gives rise to a complete classification of all possible metric forms. With respect to the algebras of the symmetry groups, this is just the Bianchi classification, but with all orientations of the orbits in the spacetime taken into account.

In every case that emerges, the commutation relations of the algebra have been solved, resulting in explicit formulae for the Killing fields, and then the Killing equations have been solved, resulting in the formulae for the metric tensors compatible with the symmetry groups considered. The degree of success in solving the Einstein equations varied very strongly from case to case. In most cases, no headway was made. In some cases the Einstein equations have been integrated either to an autonomous set of first-order equations or to a single nonlinear differential equation of second or third order. In a few cases solutions known earlier were identified in the present scheme and new invariant definitions for some of them were provided (those by Lanczos[1] and Gödel[2] will be mentioned in this paper). In three cases new solutions were found.

Since the number of cases is rather large, the results will be presented in three papers. The present paper deals with the simplest situation when two of the Killing fields are spanned on

0022-2488/98/39(1)/380/21/$15.00

velocity and rotation (the case of all three Killing fields being spanned on u and w is trivial—see Sec. VI).

The Darboux theorem was first applied as a tool for investigating the equations of motion and the Einstein equations by Plebański.[3] He showed that if a perfect fluid is rotating and isentropic while the particle number is conserved, then a similar consideration to the one presented here applies. The approach of Plebański was used by this author[4–8] to find a large collection of stationary, cylindrically symmetric solutions of Einstein's equations.

For a perfect fluid the assumptions of geodesic motion and nonzero rotation imply that the pressure is constant (see Ref. 9). Hence, from the point of view of thermodynamics, geodesic and rotating perfect fluids are isentropic and fall within the class considered by Plebański.[3] However, the approach based on the Darboux theorem applies to any timelike congruence that is of class C^1 and has zero acceleration and nonzero rotation. In particular, the velocity field of a rotating charged dust with zero Lorentz force, that was considered in several papers, has this property. The papers in which such solutions were discussed will be mentioned at the end of Sec. VIII; they are all within the same class of the classification introduced here.

In Sec. II the Darboux theorem is introduced. In Sec. III, the classification of first-order differential forms based on the Darboux theorem is applied to geodesic vector fields with rotation. When the vector field is the velocity field of a fluid, a class of preferred coordinates results, which shall be termed ''Plebański coordinates.'' In Sec. IV, by way of an example, the consideration of Sec. III is applied to the rotating dust solution of Stephani.[10] In Sec. V it is shown that each Killing vector field that might possibly exist in a rotating dust spacetime is determined by two functions of two variables. If the Killing field is not spanned on velocity and rotation, then the Plebański coordinates may be adapted to it so that it acquires the unique form $k^\alpha = \delta^\alpha_1$. The Gödel solution is used to illustrate the various forms of the Killing fields that may arise.

In Sec. VI, the consideration of Secs. III and V is applied to the situation when there exist three Killing vector fields. When all three of them are spanned on u^α and w^α, the result is trivial: the group becomes two-dimensional, and this case is not considered here. When two of them are spanned on u^α and w^α while the third one is not, two cases arise that correspond to different Bianchi types (II and I) of the groups. These are investigated in Secs. VII and VIII. The solutions of Lanczos[1] and of Gödel[2] emerge as special cases in both types. The Bianchi type II metrics are defined by a single third-order differential equation, the Bianchi type I metrics are determined by a set of autonomous first-order differential equations.

Finally, in Sec. IX, other invariant definitions are given: for the class III solution by Ozsvath[11] and for the solution of Gödel.[2] The former results from the following assumptions.

(1) The source in the Einstein equations is a geodesic, rotating perfect fluid.
(2) The rotation vector field is covariantly constant.

The Gödel solution, which is a subcase of this, emerges when it is assumed in addition that the shear of the fluid flow is zero.

So far, Bianchi-type solutions of Einstein's equations with a rotating source have been searched for and found by trial and error (often with nonperfect fluid sources, e.g., with heat-flow). The results of the present series of papers show that, in the case of a dust source at least, the number of allowed possibilities is limited. It is hoped that the results will direct further research toward better-defined targets.

II. THE CLASSIFICATION OF DIFFERENTIAL FORMS OF FIRST ORDER AND THE DARBOUX THEOREM

The Darboux theorem presented below exploits the simple fact that if a differential form q of first order is defined on an n-dimensional manifold M_n, then its domain is not necessarily n-dimensional. Two cases are well-known.

(1) If q is a perfect differential so that $q=df$, where f is a scalar function on M_n, then f can be chosen as one of the coordinates, and the form becomes one-dimensional.
(2) If q has an integrating factor so that $q=gdf$, where f and g are independent scalar functions on M_n, then f and g can be chosen as two of the coordinates, and then the domain of q is the two-dimensional (f,g)-surface.

The Darboux theorem summarizes all the cases that can occur. It is based on the following classification (see also Ref. 3)

Definition: Let q be a differential form of first order.

If $Q_{2l}:=dq\wedge\cdots\wedge dq$ (multiplied l times) $\neq 0$, but $q\wedge Q_{2l}=0$, then q is said to be of class $2l$.

If $Q_{2l+1}:=q\wedge Q_{2l}\neq 0$, but $dQ_{2l+1}\equiv dq\wedge Q_{2l}=0$, then q is said to be of class $(2l+1)$. □

Then the following holds.

The theorem of Darboux: The form q is of class $2l$ if and only if there exists a set of $2l$ independent functions $(\xi_1,\ldots,\xi_l,\eta_1,\ldots,\eta_l)$ such that

$$q=\eta_1 d\xi_1+\eta_2 d\xi_2+\cdots+\eta_l d\xi_l. \tag{2.1}$$

The form q is of class $(2l+1)$ if and only if there exists a set of $(2l+1)$ independent functions $(\tau,\xi_1,\ldots,\xi_l,\eta_1,\ldots,\eta_l)$ such that

$$q=d\tau+\eta_1 d\xi_1+\eta_2 d\xi_2+\cdots+\eta_l d\xi_l. \tag{2.2}$$

A proof of this theorem can be found in Ref. 12.

Evidently, the class of q cannot be larger than the dimension of the manifold on which q is defined. Hence, the Darboux theorem implies that in a four-dimensional spacetime V_4 any differential form of first order can be represented as

$$q=\sigma d\tau+\eta d\xi, \tag{2.3}$$

where σ, τ, η and ξ are scalar functions on V_4.

Any vector field u^α on V_4 defines the following differential form:

$$q_u:=u_\alpha dx^\alpha. \tag{2.4}$$

According to (2.3), in the most general case there exist scalar functions σ, τ, η and ξ such that

$$u_\alpha=\sigma\tau_{,\alpha}+\eta\xi_{,\alpha}. \tag{2.5}$$

Note that the functions in (2.5) are not uniquely defined. Since we shall not use (2.5) in the most general case, we shall determine the nonuniqueness only in the subcase that is of direct interest to us (see Sec. III).

For the most general case of (2.5), the four functions are independent in the sense that the Jacobian,

$$\frac{\partial(\sigma,\tau,\eta,\xi)}{\partial(x^0,x^1,x^2,x^3)}\neq 0. \tag{2.6}$$

Hence, they can be chosen as coordinates in the spacetime. In Refs. 4 and 7 it was shown that if u^α is the velocity field of an isentropic perfect fluid in which the particle number is conserved, then $\sigma=1/H$, where H is the enthalpy per one particle of the fluid, and further limitations on u_α follow from the particle number conservation. No other applications of (2.5) in the general case are known to this author.

III. GEODESICALLY MOVING FLUIDS

To any timelike vector field u_α normalized to unity (so that $u_\alpha u^\alpha = 1$) the formula from Refs. 13 and 14 may be applied:

$$u_{\alpha;\beta} = \dot{u}_\alpha u_\beta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \tfrac{1}{3}\,\theta h_{\alpha\beta}, \tag{3.1}$$

which gives rise to the well-known definitions of acceleration $\dot{u}^\alpha$, expansion θ, shear $\sigma_{\alpha\beta}$ and rotation $\omega_{\alpha\beta}$. In the signature $(+---)$ used here, the projection tensor $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta. \tag{3.2}$$

The following properties of $\dot{u}^\alpha$, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ will be useful in further considerations:

$$\dot{u}_\alpha u^\alpha = 0, \quad \sigma_{\alpha\beta} u^\beta = \omega_{\alpha\beta} u^\beta = 0. \tag{3.3}$$

We shall assume from now on that u_α is the velocity field of a fluid and that $\dot{u}_\alpha = 0$, i.e., that the particles of the fluid move on geodesics. Then, from (2.5) we have

$$\omega_{\alpha\beta} = \tfrac{l}{2}\,(u_{\alpha,\beta} - u_{\beta,\alpha}) = \tfrac{1}{2}\,(\sigma_{,\beta}\tau_{,\alpha} - \sigma_{,\alpha}\tau_{,\beta} + \eta_{,\beta}\xi_{,\alpha} - \eta_{,\alpha}\xi_{,\beta}), \tag{3.4}$$

and from (3.3) we have

$$(u^\beta \sigma_{,\beta})\,\tau_{,\alpha} - (u^\beta \tau_{,\beta})\,\sigma_{,\alpha} + (u^\beta \eta_{,\beta})\,\xi_{,\alpha} - (u^\beta \xi_{,\beta})\,\eta_{,\alpha} = 0. \tag{3.5}$$

There are two possibilities now.

I. At least one of the four scalar products in (3.5) is nonzero. In this case (3.5) implies that at most three of the functions (σ,τ,η,ξ) are independent, and so the form (2.4) will not be of class 4.

II. All the four scalar products are zero. However, this means that the gradients of (σ,τ,ξ,η) are at every point confined to the 3-space element orthogonal to u_α, i.e., that there is a functional relation among these four functions. Again, the form (2.4) cannot be of class 4.

Hence, for a geodesically moving fluid the form (2.4) is of class at most 3, i.e., at most 3 independent functions τ, η, ξ exist such that

$$u_\alpha = \tau_{,\alpha} + \eta \xi_{,\alpha}. \tag{3.6}$$

From here on, the reasoning used in Refs. 3 and 4 applies almost unchanged. With (3.6) we have in (3.4),

$$\omega_{\alpha\beta} = \tfrac{1}{2}\,(\eta_{,\beta}\xi_{,\alpha} - \eta_{,\alpha}\xi_{,\beta}), \tag{3.7}$$

and in (3.5),

$$(u^\beta \eta_{,\beta})\,\xi_{,\alpha} - (u^\beta \xi_{,\beta})\,\eta_{,\alpha} = 0. \tag{3.8}$$

There are again two possibilities.

I. Either $(u^\beta \eta_{,\beta})$ and $(u^\beta \xi_{,\beta})$ do not vanish simultaneously, and then (3.8) implies that η and ξ are functionally related, in which case (3.6) implies that u_α is a gradient of a function, and so $\omega_{\alpha\beta} \equiv 0$.

II. Or ξ and η are not functionally related, in which case

$$u^\beta \xi_{,\beta} = u^\beta \eta_{,\beta} = 0, \tag{3.9}$$

and $\omega_{\alpha\beta} \neq 0$. We shall be interested only in the second case. The functions $\{\tau,\xi,\eta\}$ in (3.6) are determined up to the following transformations:

$$\xi = F(\xi', \eta'), \quad \eta = G(\xi', \eta'), \quad \tau = \tau' - S(\xi', \eta'), \tag{3.10}$$

where the functions F and G must obey the equation

$$F_{,\xi'} G_{,\eta'} - F_{,\eta'} G_{,\xi'} = 1, \tag{3.11}$$

and then S is determined by

$$S_{,\xi'} = G F_{,\xi'} - \eta', \quad S_{,\eta'} = G F_{,\eta'}. \tag{3.12}$$

Equation (3.11) is the integrability condition of Eqs. (3.12) and it ensures that the Jacobian of the transformation (3.10) equals 1. It follows that one of the functions $\{F, G\}$ can be chosen arbitrarily, the other one is then determined by (3.11) and S is fixed up to an additive constant by (3.12). The inverse transformation to (3.10) is of exactly the same form, with the corresponding F, G and S obeying (3.11) and (3.12).

Let us now make the additional assumption that the number of particles of the fluid is conserved, i.e.,

$$(\sqrt{-g} n u^\alpha)_{,\alpha} = 0, \tag{3.13}$$

where g is the determinant of the metric tensor and n is the particle number density. This equation is a necessary and sufficient condition for the existence of a function ζ such that

$$\sqrt{-g} n u^\alpha = \epsilon^{\alpha\beta\gamma\delta} \xi_{,\beta} \eta_{,\gamma} \zeta_{,\delta}. \tag{3.14}$$

Note that (3.6) and (3.9) imply that

$$u^\alpha \tau_{,\alpha} = 1, \tag{3.15}$$

and then Eq. (3.14) implies that

$$\epsilon^{\alpha\beta\gamma\delta} \tau_{,\alpha} \xi_{,\beta} \eta_{,\gamma} \zeta_{,\delta} \equiv \frac{\partial(\tau, \eta, \xi, \zeta)}{\partial(x^0, x^1, x^2, x^3)} = \sqrt{-g} n \neq 0. \tag{3.16}$$

Equation (3.14) implies also that

$$u^\alpha \zeta_{,\alpha} = 0. \tag{3.17}$$

The function ζ is determined by (3.14) up to the transformations

$$\zeta = \zeta' + T(\xi', \eta'), \tag{3.18}$$

where T is an arbitrary function. Equation (3.16) certifies that $\{\tau, \xi, \eta, \zeta\}$ can be used as coordinates in the spacetime. If they are chosen as the $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ coordinates, respectively, then Eq. (3.6) implies

$$u_0 = 1, \quad u_1 = y, \quad u_2 = u_3 = 0. \tag{3.19}$$

We will use these coordinates throughout the remaining part of the paper and call them ''Plebański coordinates.'' Equation (3.16) implies now that

$$g = -n^{-2}, \tag{3.20}$$

and Eq. (3.14) implies

$$u^\alpha = \delta^\alpha_0, \tag{3.21}$$

i.e., the Plebański coordinates are comoving. The rotation vector defined by

$$w^{\alpha} = -(1/\sqrt{-g})\epsilon^{\alpha\beta\gamma\delta}u_{\beta}u_{\gamma,\delta}, \tag{3.22}$$

assumes the form

$$w^{\alpha} = n\,\delta_3^{\alpha}. \tag{3.23}$$

Equations (3.19) and (3.21) imply that

$$g_{00} = 1, \quad g_{01} = y, \quad g_{02} = g_{03} = 0, \tag{3.24}$$

and also that the only nonvanishing components of the rotation tensor are

$$\omega_{12} = -\omega_{21} = 1/2. \tag{3.25}$$

Note that, in contrast to Refs. 3 and 4 where barotropic perfect fluids were considered, we have not assumed anything about the form of the energy–momentum tensor so far.

If we now assume that the fluid is perfect, then we conclude from the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ that either $\omega = 0$ or $p = \text{const}$ (see also Ref. 9). This means that a geodesic perfect fluid can be rotating only if it is in fact dust; the constant p can be reinterpreted as the cosmological constant. In this case, the energy–density obeys the conservation equation $(\sqrt{-g}\epsilon u^{\alpha})_{,\alpha} = 0$ and Eq. (3.13) need not be assumed separately. [For dust, results closely analogous to (3.19)–(3.25) were obtained by Ellis,[15] by adapting an orthonormal vector basis and a coordinate system to u^{α} and w^{α}. Of the exact solutions with nonzero rotation found by Ellis most, but not all, do belong to the collection considered in this series of papers. They will be described in paper 2.]

IV. EXAMPLE: THE STEPHANI SOLUTION[10]

The Stephani metric with $p = \text{const} \neq 0$ [Eq. (4.22) in Ref. 10] is not in fact a perfect fluid solution, as was found out while trying to construct the Plebański coordinates for it. (The error is deeply hidden and so far could not be corrected. I am grateful to H. Stephani for cooperation on this point.) Therefore, we shall consider only the dust solution, Eq. (4.8) in Ref. 10. In the original notation except for the signature, the solution is

$$ds^2 = \eta_{ab}dx^a dx^b - N^2(dx^1)^2, \tag{4.1}$$

where $a,b = 0,2,3$, $\eta_{ab} = \text{diag}(1,-1,-1)$ and

$$N := \tfrac{1}{2} M \ln T + g_a x^a + h,$$

$$T^2 := \eta_{ab}(x^a - f^a)(x^b - f^b), \tag{4.2}$$

the functions $M(x^1)$, $f^a(x^1)$, $g_a(x^1)$, and $h(x^1)$ all being arbitrary. The velocity field and the energy–density of the dust are, respectively,

$$u_a = T_{,a}, \quad u_1 = 0, \tag{4.3}$$

$$\kappa\epsilon = M/(NT^2). \tag{4.4}$$

The formula for the velocity field can be written as

$$u_{\alpha} = T_{,\alpha} - T_{,1}x^1{}_{,\alpha}, \tag{4.5}$$

which immediately suggests the choice of the Plebański coordinates of Sec. III:

$$\tau = T, \quad \xi = x^1, \quad \eta = -T_{,1}. \tag{4.6}$$

Equation (3.14) defining ζ is here

$$MT^{-2}\eta^{ab}T_{,b} = -\epsilon^{a1cd}T_{,1c}\zeta_{,d}\,. \tag{4.7}$$

The following identity is useful in calculations:

$$T_{,0}{}^2 - T_{,2}{}^2 - T_{,3}{}^2 = 1. \tag{4.8}$$

Using this, one can verify that only two of the three equations (4.7) are independent. The solution of (4.7) is

$$\zeta = M[U + \lambda(T_{,1})], \tag{4.9}$$

where $\lambda(T_{,1})$ is an arbitrary function, and U is defined by

$$U = \int [f^3{}_{,1}(1-X^2) - f^0{}_{,1}Y(X) + f^2{}_{,1}XY(X)]^{-1}dX, \tag{4.10}$$

the function $Y(X)$ being determined by

$$T_{,1}(1-X^2-Y^2)^{1/2} = -f^0{}_{,1} + f^2{}_{,1}X + f^3{}_{,1}Y. \tag{4.11}$$

In the integral (4.10), the coordinate x^1 and the quantity $T_{,1}$ are to be treated as parameters independent of X. After the integral is calculated, one should substitute for $T_{,1}$ from (4.11), while X and Y are to be replaced by

$$X = (x^2 - f^2)/(x^0 - f^0), \quad Y = (x^3 - f^3)/(x^0 - f^0). \tag{4.12}$$

The integral in (4.10) is expressible in terms of elementary functions, but the result is rather complicated.

As can be seen, the solution (4.1)–(4.4) becomes very complicated in the Plebański coordinates and it is unlikely that it could be found in such a form from the Einstein equations. Thus, although the coordinates are invariantly defined, they are not necessarily convenient for explicit calculations.

A collection of other solutions represented in the Plebański coordinates can be found in the extended version of Ref. 6.

V. THE KILLING VECTOR FIELDS COMPATIBLE WITH ROTATION

We shall assume that the symmetries of the spacetime (if any exist) are inherited by the source, i.e., that if the Lie derivative of the metric tensor $g_{\alpha\beta}$ along the vector field k^α is zero, $\mathscr{L}_k g_{\alpha\beta} = 0$, then the velocity field and the particle number density are also invariant: $\mathscr{L}_k u^\alpha = 0 = \mathscr{L}_k n$. (For a pure perfect fluid source the inheritance is guaranteed.) It follows that the rotation tensor must also be invariant, $\mathscr{L}_k \omega_{\alpha\beta} = 0$.

In consequence of (3.21) the equation $\mathscr{L}_k u^\alpha \equiv k^\mu u^\alpha{}_{,\mu} - u^\mu k^\alpha{}_{,\mu} = 0$ implies that

$$k^\alpha{}_{,t} = 0. \tag{5.1}$$

In consequence of (3.23) and of the assumption $\mathscr{L}_k n \equiv k^\alpha n_{,\alpha} = 0$, the equation $\mathscr{L}_k w^\alpha = 0$ implies

$$k^\alpha{}_{,z} = 0. \tag{5.2}$$

The equation $\mathscr{L}_k \omega_{\alpha\beta} = 0$, in consequence of (3.25) implies

$$k^1{}_{,x} + k^2{}_{,y} = 0, \tag{5.3}$$

and the equation $\mathcal{L}_k u_\alpha = 0$, in consequence of (3.19) implies

$$k^0{}_{,x} = -k^2 - yk^1{}_{,x}, \quad k^0{}_{,y} = -yk^1{}_{,y}. \tag{5.4}$$

(The equations $\mathcal{L}_k u_\alpha = 0$ and $\mathcal{L}_k u^\alpha = 0$ provide independent pieces of information because the equations $\mathcal{L}_k g_{\alpha\beta} = 0$ have not been used so far.) Equation (5.3) is the integrability condition of Eqs. (5.4). The general solution of Eqs. (5.1)–(5.4) is

$$k^0 = C + \phi - y\phi_{,y}, \quad k^1 = \phi_{,y}, \quad k^2 = -\phi_{,x}, \quad k^3 = \lambda, \tag{5.5}$$

where $\phi(x,y)$ and $\lambda(x,y)$ are arbitrary functions and C is an arbitrary constant. Symmetries need not be present, in fact the Stephani[10] solution considered in Sec. IV is an example of a rotating dust solution with no symmetries. In this case $\phi = \lambda = C = 0$. However, if any symmetries are present, then the Killing vector fields must have the form (5.5).

Suppose that ϕ is not a constant, i.e., that a Killing vector field k^α exists that has a nonzero component in the x- or y-direction (in invariant terms this means that the vector field k^α is not spanned on the vector fields of velocity, u^α, and rotation, w^α). We can then, within the Plebański class defined in Sec. III, adapt the coordinates to k^α in such a way that $k^{\alpha'} = \delta^{\alpha'}_1$, i.e., so that the metric becomes independent of x'. From (3.10)–(3.12) and (3.18) the transformation functions are

$$t' = t - S(x,y), \quad x' = F(x,y), \quad y' = G(x,y), \quad z' = z + T(x,y), \tag{5.6}$$

where T is arbitrary, while F, G, and S obey

$$F_{,x}G_{,y} - F_{,y}G_{,x} = 1, \quad S_{,x} = GF_{,x} - y, \quad S_{,y} = GF_{,y}. \tag{5.7}$$

In order to lead to $k^{\alpha'} = \delta^{\alpha'}_1$ the functions F, G, and T must obey in addition

$$-(\phi + C) + GF_{,x}\phi_{,y} - GF_{,y}\phi_{,x} = 0,$$

$$F_{,x}\phi_{,y} - F_{,y}\phi_{,x} = 1, \quad G_{,x}\phi_{,y} - G_{,y}\phi_{,x} = 0, \tag{5.8}$$

$$T_{,x}\phi_{,y} - T_{,y}\phi_{,x} = -\lambda. \tag{5.9}$$

The unique solution of Eqs. (5.8) is $G = \phi + C$, which obeys (5.7) as well [in virtue of the second of (5.8)]. Equation (5.9) simply defines the accompanying T which is seen to exist always. Since ϕ was assumed nonconstant, the transformation is nonsingular (in fact its Jacobian equals just 1), and results in $\phi = y$ in the new coordinates. As already noticed, the metric becomes independent of x after the transformation. This property is preserved by the transformations (5.6), but with F, G, S, and T restricted now by

$$G = y, \quad F = x + H(y), \quad T = T(y), \quad S = \int yH_{,y}dy + A, \tag{5.10}$$

where A is an arbitrary constant and H, T are arbitrary functions. The functions given by (5.10) fulfill (5.7) identically. Note that the transformation to $k^{\alpha'} = \delta^{\alpha'}_1$ exists irrespectively of any possible functional relation among λ, ϕ, and C in (5.5); the only case when it fails is $\phi = \text{const}$.

A solution of the Einstein equations may have more than one Killing vector field of the form (5.5). In that case, the transformation (5.8)–(5.9) changes only one of them to the preferred form, the others will preserve their more complicated appearance. An example of this situation is the Gödel solution[2] transformed to the Plebański coordinates; see Refs. 4 and 5:

$$ds^2 = (dt + ydx)^2 - \tfrac{1}{2}y^2dx^2 - (\kappa\epsilon y^2)^{-1}dy^2 - 2\kappa\epsilon^{-1}dz^2, \tag{5.11}$$

where $\kappa = 8\pi G/c^4$ and ϵ is the energy–density related to the cosmological constant by

$$\Lambda = \tfrac{1}{2}\,\kappa\epsilon. \tag{5.12}$$

(Note that if the cosmological constant is reinterpreted as pressure, then the resulting perfect fluid has the equation of state $\epsilon = p$. Hence, the Gödel solution may have been the first example considered in the literature of a ''stiff perfect fluid,'' now familiar from the studies of solution-generating techniques; see, e.g., Verdaguer.[16]) The symmetry group of this solution is 5-dimensional, the independent 1-parametric subgroups were given in Ref. 4. Those connected with nonconstant ϕ in (5.5) are the following three:

$$x = x' + s_1, \tag{5.13}$$

$$x = e^{-s_2}x', \quad y = e^{s_2}y', \tag{5.14}$$

$$t = t' + (2\sqrt{2}/K)\arctan[\sqrt{2}s_3(Ky')^{-1}(1 - s_3x')^{-1}],$$

$$x = [K^2x'y'^2(1 - s_3x') - 2s_3]/[2s_3^2 + K^2y'^2(1 - s_3x')^2], \tag{5.15}$$

$$y = (1 - s_3x')^2y' + 2s_3^2/(K^2y'),$$

where s_1, s_2, and s_3 are the group parameters and $K := (\kappa\epsilon)^{1/2}$. The Killing vectors are, respectively, $k^\alpha_{(1)} = \delta^\alpha_1$ [corresponding to $C = \lambda = 0$, $\phi = y$, the one constructed in (5.8)–(5.9)], $k^\alpha_{(2)} = -x\,\delta^\alpha_1 + y\,\delta^\alpha_2$ (corresponding to $\phi = -xy$) and $k^\alpha_{(3)} = 4(K^2y)^{-1}\delta^\alpha_0 + [x^2 - 2/(Ky)^2]\delta^\alpha_1 - 2xy\,\delta^\alpha_2$ [corresponding to $\phi = x^2y + 2/(K^2y)$].

VI. THE ALGEBRA OF THREE KILLING FIELDS

Suppose that three Killing vector fields exist and all three are spanned on u^α and w^α, so that $\phi = \text{const}$ in (5.5) for each of them, i.e.,

$$k^\alpha_{(i)} = C_i\delta^\alpha_0 + \lambda_i(x,y)\,\delta^\alpha_3, \quad i = 1,2,3. \tag{6.1}$$

From the Killing equations one can then easily conclude that constants α_1, α_2, and α_3 exist such that $\alpha_1 k_{(1)} + \alpha_2 k_{(2)} + \alpha_3 k_{(3)} = 0$, i.e., the symmetry group is in fact two-dimensional. Hence, no three-dimensional symmetry group with the generators (6.1) exists; for a three-dimensional group at least one of the generators must be linearly independent of u^α and w^α at every point of the spacetime region under consideration. [The algebra (6.1) corresponds to a three-dimensional group that has two-dimensional orbits, and it turns out that in the case considered the group has to be two-dimensional as well. As will follow from the whole of the present work, three-dimensional symmetry groups with two-dimensional orbits just do not exist for rotating dust.]

In Secs. VII and VIII we shall consider the situation when exactly one generator, $k^\alpha_{(1)}$, is everywhere linearly independent of u^α and w^α, while the other two, $k^\alpha_{(2)}$ and $k^\alpha_{(3)}$, are of the form (6.1). In agreement with the result of Sec. V, the Plebański coordinates can be adapted to $k^\alpha_{(1)}$ so that

$$k^\alpha_{(1)} = \delta^\alpha_1, \tag{6.2}$$

while

$$k^\alpha_{(2)} = C_2\delta^\alpha_0 + \lambda_2(x,y)\,\delta^\alpha_3, \quad k^\alpha_{(3)} = C_3\delta^\alpha_0 + \lambda_3(x,y)\,\delta^\alpha_3, \tag{6.3}$$

and the coordinate transformations preserving (6.2) and (6.3) are (5.10). Note that C_2 and C_3 cannot vanish simultaneously because otherwise the Killing equations immediately imply that either $k^\alpha_{(3)}=\text{const}\, k^\alpha_{(2)}$ (in which case the symmetry group is two-dimensional) or the metric is singular. However, with no loss of generality we can assume that

$$C_2 \neq 0 = C_3, \tag{6.4}$$

because the Killing vector fields are determined up to linear combinations among them. Hence, if initially $C_2 \neq 0 \neq C_3$, then we take $k'^\alpha_{(3)} = k^\alpha_{(3)} - (C_3/C_2)k^\alpha_{(2)}$ instead of $k^\alpha_{(3)}$ as the basis generator. If initially $C_2 = 0 \neq C_3$, then we exchange the labels ''2'' and ''3.''

We shall keep the choice (6.4) throughout Secs. VII and VIII.

VII. THE GENERATORS, THE KILLING EQUATIONS AND THE EINSTEIN EQUATIONS FOR A BIANCHI TYPE II ALGEBRA

All the cases that arise follow as limits at different stages of calculation from the generic case $\lambda_2 \neq 0$, and we shall consider it first. The commutators of the Killing vectors are

$$\begin{gathered} [k_{(1)}, k_{(2)}]^\alpha = (\lambda_{2,x}/\lambda_3) k^\alpha_{(3)}, \quad [k_{(2)}, k_{(3)}]^\alpha = 0, \\ [k_{(1)}, k_{(3)}]^\alpha = (\lambda_{3,x}/\lambda_3) k^\alpha_{(3)}. \end{gathered} \tag{7.1}$$

The Killing vector fields will thus form a Lie algebra when

$$\lambda_{2,x} = b\lambda_3, \quad \lambda_{3,x} = c\lambda_3, \tag{7.2}$$

where b and c are arbitrary constants. The case $c \neq 0$ has to be considered separately. Then

$$\lambda_3 = \beta(y) e^{cx}, \quad \lambda_2 = (b/c)\beta(y) e^{cx} + \alpha(y), \tag{7.3}$$

where $\alpha(y)$ and $\beta(y)$ are arbitrary functions. However, in this case we can take $k'^\alpha_{(2)} = k^\alpha_{(2)} - (b/c)k^\alpha_{(3)}$ as the new basis generator instead of $k^\alpha_{(2)}$, and the result is equivalent to assuming $b=0$. Hence, with $c \neq 0$, we can take $b=0$ with no loss of generality.

The further procedure consists of the following steps.

(1) Adapt the coordinates to the Killing fields to make them as simple as possible.
(2) Solve the Killing equations for the components of the metric tensor.
(3) Simplify the metric as far as possible by coordinate transformations.
(4) Solve the Einstein equations.

The coordinate transformations in steps 1 and 3 in general lead out of the Plebański class defined in Sec. III.

This procedure will be presented in some detail below. In the present case the result from the Einstein equations is that either $c=0$ or there is no rotation. Since we are interested in rotating solutions only, the case $c \neq 0 = b$ need not be followed further. We thus assume

$$c = 0. \tag{7.4}$$

Then

$$\lambda_3 = \beta(y), \quad \lambda_2 = b\beta(y)x + \alpha(y). \tag{7.5}$$

The algebra of the Killing vector fields is of Bianchi type II when $b \neq 0$ and of Bianchi type I when $b=0$.

In order to simplify the Killing vectors we now transform the coordinates as follows:

$$(t',x',y')=(t,x,y), \quad z'=-(\alpha/C_2)t+z/\beta. \tag{7.6}$$

The transformation is not of the form (5.10), so the new coordinates do not belong to the Plebański class, and the forms of velocity, rotation, and the metric will no longer agree with (3.19)–(3.25). The Killing vector fields in the new coordinates become

$$k_{(1)}^{\alpha}=\delta_1^{\alpha}, \quad k_{(2)}^{\alpha}=\delta_0^{\alpha}+bx\,\delta_3^{\alpha}, \quad k_{(3)}^{\alpha}=\delta_3^{\alpha}, \tag{7.7}$$

while the velocity and rotation fields become

$$u^{\alpha}=\delta_0^{\alpha}-(\alpha/C_2)\,\delta_3^{\alpha}, \quad w^{\alpha}=(n/\beta)\,\delta_3^{\alpha}. \tag{7.8}$$

The transformed metric is independent of x and z, while the Killing equations for $k_{(2)}^{\alpha}$ imply

$$\begin{gathered}
g_{00}=1+(\alpha/C_2)^2h_{33}(y), \quad g_{01}=y+(\alpha/C_2)g_{13}, \\
g_{02}=0, \quad g_{03}=(\alpha/C_2)h_{33}(y), \\
g_{11}=h_{33}(y)(bt)^2-2h_{13}(y)bt+h_{11}(y), \\
g_{12}=-h_{23}(y)bt+h_{12}(y), \quad g_{13}=-h_{33}(y)bt+h_{13}(y), \\
g_{22}=h_{22}(y), \quad g_{23}=h_{23}(y), \quad g_{33}=h_{33}(y),
\end{gathered} \tag{7.9}$$

where $h_{ij}(y)$, $i,j=1,2,3$, are arbitrary functions of y, to be found from the Einstein equations.

The orbits of the symmetry group are now the hypersurfaces $y=\text{const}$. In order to follow the standard technique of the Bianchi-type spaces we should now carry out a coordinate transformation that preserves (7.7) and makes the y-coordinate curves orthogonal to the group orbits, so that $g'_{02}=g'_{12}=g'_{23}=0$ after the transformation. This step is not in fact necessary for solving the Einstein equations (in general it only reshuffles the unknown functions without eliminating any of them), but in the case under consideration it leads to a simplification. The transformation is

$$\begin{gathered}
t=t'+f_0(y'), \quad x=x'+f_1(y'), \quad y=f_2(y'), \\
z=z'+bf_1(y')t'+f_3(y'),
\end{gathered} \tag{7.10}$$

where $f_{\alpha}(y')$ must obey

$$\begin{gathered}
f_{0,y'}=-f_2f_{1,y'}+(\alpha/C_2)h_{23}f_{2,y'}, \\
(-f_2^2+h_{11}-h_{13}^2/h_{33})f_{1,y'}+(h_{12}-h_{13}h_{23}/h_{33}+\alpha f_2h_{23}/C_2)f_{2,y'}=0, \\
f_{3,y'}=-(\alpha/C_2)f_{0,y'}+(bf_0-h_{13}/h_{33})f_{1,y'}-(h_{23}/h_{33})f_{2,y'}.
\end{gathered} \tag{7.11}$$

The equations are well-defined because of the following.

I. $h_{33}\neq 0$; otherwise the rotation vector would be null, which is a physical impossibility.

II. $-f_2^2+h_{11}-h_{13}^2/h_{33}\neq 0$; otherwise the determinant of the metric tensor becomes positive; i.e., the metric acquires an unphysical signature.

Equations (7.11) are to be understood as follows. The function $f_2(y')$ can be chosen arbitrarily, therefore we choose it so that $g'_{22}=-1$ after the transformation. With $f_2(y')$ thus chosen, $f_1(y')$ is found from the second of (7.11), then $f_0(y')$ is found from the first of (7.11), and finally $f_3(y')$ is found from the third of (7.11).

After the transformation the metric becomes (primes dropped)

$$
\begin{aligned}
g_{00}&=1+(\alpha/C_2+bf_1)^2h_{33},\\
g_{01}&=Y(y)+(\alpha/C_2+bf_1)(-h_{33}bt+h_{13}-bf_0h_{33}),\\
g_{02}&=g_{12}=g_{23}=0,\\
g_{03}&=(\alpha/C_2+bf_1)h_{33},\\
g_{11}&=h_{33}b^2(t+f_0)^2-2h_{13}b(t+f_0)+h_{11},\\
g_{13}&=-h_{33}b(t+f_0)+h_{13},\\
g_{22}&=-1,\quad g_{33}=h_{33}(y),
\end{aligned}
\tag{7.12}
$$

where $h_{ij}(y)$, $i,j=1,2,3$, $f_0(y),f_1(y)$, $Y(y)=f_2(y)$ and $\alpha(y)$ are functions to be found from the Einstein equations, and b and C_2 are arbitrary constants, $C_2\neq 0$.

For convenience in calculations we introduce the new functions $G(y)$, $A(y)$, $k_{13}(y)$, and $F(y)$ by

$$
\begin{gathered}
g_{33}=-G^2(y),\quad \alpha/C_2+bf_1=A(y),\quad h_{13}=-G^2(k_{13}+bf_0),\\
h_{11}=Y^2-F^2-k_{13}^2G^2+b^2G^2f_0^2-2bf_0k_{13}G^2.
\end{gathered}
\tag{7.13}
$$

The velocity field in the coordinates of (7.12)–(7.13) is

$$
u^{\alpha}=\delta_0^{\alpha}-A\delta_3^{\alpha}.
\tag{7.14}
$$

Now the metric form is:

$$
ds^2=(dt+Ydx)^2-(Fdx)^2-dy^2-G^2[Adt-(bt-k_{13})dx+dz]^2.
\tag{7.15}
$$

The components of the Einstein tensor will be referred to the orthonormal tetrad of forms $e^i=e^i{}_{\alpha}dx^{\alpha}$, $i=0,1,2,3$, uniquely implied by (7.15). Note that $e^0=u_{\alpha}dx^{\alpha}$. Hence, the Einstein equations are

$$
\begin{gathered}
G_{00}=\kappa\epsilon,\\
G_{11}=G_{22}=G_{33}=\Lambda,\quad G_{ij}=0,\quad \text{when}\ \ i\neq j,
\end{gathered}
\tag{7.16}
$$

where ϵ is the energy–density and Λ is the cosmological constant.

The equation $G_{12}=0$ implies that $bA_{,y}=0$. The case $b=0$ will be considered separately below, so we take here

$$
A=\text{const.}
\tag{7.17}
$$

Then $G_{02}=0$ implies

$$
k_{13}=\text{const.}
\tag{7.18}
$$

We can then carry out the coordinate transformation:

$$
z=z'-At-k_{13}x,\quad (t,x,y)=(t',x',y'),
\tag{7.19}
$$

which has the same result as if

$$A=k_{13}=0, \tag{7.20}$$

and we shall assume this from now on. The metric is still independent of x and of z, while $A=k_{13}=0$ implies $g_{13}=0$, i.e., the Killing vectors $k_{(1)}^{\alpha}=\delta_1^{\alpha}$ and $k_{(3)}^{\alpha}=\delta_3^{\alpha}$ are orthogonal to each other. The equation $G_{01}=0$ then has the integral

$$Y_{,y}G/F=B=\text{const}, \tag{7.21}$$

and we can assume $B\neq 0$ because rotation would be zero with $B=0=Y_{,y}$.

At this point, only the diagonal components of the Einstein tensor are still nonzero, of which G_{00} just defines the energy–density, and the other three are functionally dependent (i.e., if $G_{11}=\Lambda=G_{22}$ are fulfilled, then so is $G_{33}=\Lambda$). They determine $F(y)$ and $G(y)$.

It is convenient to introduce $Y(y)$ as the new variable. The equation $G_{11}+G_{22}=2\Lambda$ can then be written, with the help of (7.21), as

$$(F^2G_{,Y}/G)_{,Y}=2\Lambda G^2/B^2-\tfrac{1}{2}, \tag{7.22}$$

and so

$$F^2=\left(C-\frac{1}{2}\,Y+2\,\frac{\Lambda}{B^2}\int G^2dY\right)G/G_{,Y}, \tag{7.23}$$

where C is a new arbitrary constant (we can assume $G_{,Y}\neq 0$ because $G_{,Y}=0$ implies $b=0$ from $G_{11}-G_{22}=0$, and $b=0$ will be considered separately). Using (7.23) in $G_{22}=\Lambda$ we obtain the following integro-differential equation that determines G:

$$-\frac{1}{4}\,b^2GG_{,Y}+\frac{1}{2}\,(B/G)^2\left(C-\frac{1}{2}\,Y+2\,\frac{\Lambda}{B^2}\int G^2dY\right)^2(G_{,Y}/G-G_{,YY}/G_{,Y})=0. \tag{7.24}$$

In the special case $\Lambda=0$ this becomes an ordinary second-order differential equation. It is easy to get rid of the integral by transforming (7.24) appropriately and differentiating the result by Y [in this way a third-order differential equation for $G(Y)$ is obtained] or by introducing the new variable $u(Y)$ by $dY/du=1/G^2$ [this results in a second-order equation for $G(u)$]. However, no progress toward solving (7.24) results in either case.

With the help of the equations $G_{11}=\Lambda=G_{22}$ the formula for energy–density may be simplified to

$$\kappa\epsilon=(B/G)^2-(bG)^2-2\Lambda. \tag{7.25}$$

Note that the solutions considered here have a meaningful limit $b=0$.

When $G=\text{const}$, Eqs. (7.23) and (7.24) no longer apply and one has to go back to the Einstein equations. They imply

$$G^2=B^2/(4\Lambda) \tag{7.26}$$

(i.e., necessarily $\Lambda>0$) and

$$F^2=\tfrac{1}{2}\,Y^2+DY+E, \tag{7.27}$$

where D and E are constants. If Y is chosen as the new coordinate in place of y, then from (7.21) and (7.26) the metric component g_{YY} is

$$g_{YY}=-(G/BF)^2=-1/(4\Lambda F^2), \tag{7.28}$$

and the resulting metric is the Gödel solution (see Ref. 4). Note that $G=\text{const}$ is equivalent to $\epsilon=\text{const}$; see Eq. (7.25).

When $G_{,Y}\neq 0=b$, Eq. (7.24) implies $G=e^{DY+E}$, and this leads to the Lanczos solution (see Ref. 4).

These derivations of the Lanczos and Gödel solutions lead to their invariant definitions that are based on weaker assumptions than the definitions known so far. The definitions are:

(1) The source in the Einstein equations is a rotating dust.
(2) The spacetime has a 3-dimensional symmetry group.
(3) Two of the symmetry generators are spanned on the vector fields of velocity u^α and rotation w^α, while the third one is linearly independent of u^α and w^α at every point.
(4) The generators form a Bianchi type II algebra.
(5) In the solutions of the Einstein equations, the Bianchi type I limit is taken of the Bianchi type II symmetry.
(6) The Gödel solution results when the matter–density is constant; the Lanczos solution results when the density is not constant.

The generalization with respect to the earlier definition is contained in point 3: in previous derivations the two generators were assumed to be collinear with u^α and w^α, respectively, from the beginning.

VIII. THE GENERATORS, THE KILLING EQUATIONS AND THE EINSTEIN EQUATIONS FOR A BIANCHI TYPE I ALGEBRA

We shall consider the case $b=c=0$ in (7.1)–(7.2). The reasoning up to Eq. (7.16) applies also here, but (7.17) no longer follows. Instead, the equation $G_{13}=0$ can be integrated with the result

$$k_{13,y}=BF/G^3-YA_{,y}, \tag{8.1}$$

where B is an arbitrary constant; the equation $G_{01}=0$ can be integrated to

$$Y_{,y}=(C-BA)F/G, \tag{8.2}$$

where C is an arbitrary constant; and the equation $G_{03}=0$ can be integrated to

$$A_{,y}=(BY-D)/(FG^3), \tag{8.3}$$

where D is one more arbitrary constant.

At this point, only the diagonal components of the Einstein tensor survive, and $G_{00}=\kappa\epsilon-\Lambda$ just defines the energy–density. The equations $G_{11}=\Lambda=G_{22}=G_{33}$ can be written as

$$-\frac{B}{4FG}\,k_{13,y}+\frac{C-BA}{4FG}\,Y_{,y}+G_{,yy}/G-\frac{2BY-D}{4FG}\,A_{,y}=\Lambda, \tag{8.4}$$

$$-\frac{B}{4FG}\,k_{13,y}+\frac{C-BA}{4FG}\,Y_{,y}+\frac{F_{,y}G_{,y}}{FG}-\frac{D}{4FG}\,A_{,y}=\Lambda, \tag{8.5}$$

$$\frac{3B}{4FG}\,k_{13,y}-\frac{C-BA}{4FG}\,Y_{,y}+F_{,yy}/F+\frac{3D}{4FG}\,A_{,y}=\Lambda. \tag{8.6}$$

[In order to arrive at this form, one has to calculate B from (8.1) and replace one factor B in B^2 by the resulting expression; then replace one $Y_{,y}$ in $Y_{,y}^2$ from (8.2) and replace one $A_{,y}$ in $A_{,y}^2$ from

(8.3).] The set (8.4)–(8.6) can be integrated to a first-order set. Subtracting (8.6) from (8.4) and multiplying the result by FG we obtain an equation that is easily integrated to

$$FG_{,y}-GF_{,y}-Bk_{13}-\tfrac{1}{2}BAY+\tfrac{1}{2}CY-\tfrac{1}{2}DA=E=\text{const.} \tag{8.7}$$

Now adding (8.3) and (8.4), and multiplying the result by FG we obtain another integrable equation whose integral can be written in the form

$$FG_{,y}=\frac{1}{2}Bk_{13}+\frac{1}{2}BAY-\frac{1}{2}CY+2\Lambda\int FG\,dy+H_0, \tag{8.8}$$

where H_0 is an arbitrary constant. The integral can be calculated if the new variable $u(y)$ is introduced by

$$dy/du=1/(FG). \tag{8.9}$$

From (8.7) and (8.8) it follows that

$$GF_{,y}=-\frac{1}{2}Bk_{13}-\frac{1}{2}DA-E+2\Lambda\int FG\,dy+H_0. \tag{8.10}$$

In the set (8.4)–(8.6) there remains one equation that has still not been used. However, at this point it merely introduces a relation between the arbitrary constants, i.e., implicitly defines H_0 in terms of the other constants. This is seen as follows: substitute for $k_{13,y}$, $Y_{,y}$, $F_{,y}$, $G_{,y}$, and $A_{,y}$ from (8.1)–(8.3), (8.8), and (8.10) in (8.5), thereby obtaining an algebraic equation (i.e., one without derivatives). Differentiate it by y and eliminate the derivatives in the same way again. What results is an identity $0=0$. Hence, the left-hand side of (8.5) is identically constant in virtue of the other equations.

In terms of the variable u from (8.9), Eqs. (8.1)–(8.3), (8.8), and (8.10) form an autonomous set of first-order equations that can be investigated further by qualitative methods (see, e.g., Ref. 17). This is left as a subject for a separate study.

In analogy with the Bianchi type I spatially homogeneous (nonrotating) dust solutions (see Ref. 18) one might expect further progress by adapting the coordinates suitably (in the case considered in Ref. 18, the metric can be diagonalized). However, this author was not able to achieve any such progress.

The functions $A(y)$ and $k_{13}(y)$ have invariant meaning: they are proportional to the scalar products of the Killing vectors [see Eqs. (7.7) and (7.15) with $b=0$]:

$$A=-g_{\alpha\beta}k^{\alpha}_{(2)}k^{\beta}_{(3)}/G^2, \quad k_{13}=-g_{\alpha\beta}k^{\alpha}_{(1)}k^{\beta}_{(3)}/G^2 \tag{8.11}$$

(note that $G^2=-g_{\alpha\beta}k^{\alpha}_{(3)}k^{\beta}_{(3)}$, i.e., it is a scalar, too). Hence, $A=0$ and $k_{13}=0$ are invariant properties. Note that $A=0$ implies, through (8.3), that either $Y=\text{const}$ (in which case there is no rotation) or $B=D=0$. In the latter case, $k_{13}=\text{const}$ and the coordinate transformation $z=z'-k_{13}x$ leads to $k_{13}=0$ in the new coordinates. With $A=k_{13}=0$, the Lanczos and Gödel models result from the Einstein equations as the only solutions. Hence, one more invariant definition of these models follows, similar to the six-point definition at the end of Sec. VII. Points 1, 2, 3, and 6 remain unchanged, while points 4 and 5 are replaced by the following.

(4) The generators form a Bianchi type I algebra.

(5) From the two generators spanned on u^{α} and w^{α}, two linear combinations can be constructed that are orthogonal to each other.

Point 5′ is equivalent to the existence of coordinates in which $A=0$.

Note that the Bianchi type I models considered in this section are more general than the Bianchi type I limit of the models from Sec. VII; those from Sec. VII had $A=k_{13}=0$ in virtue of Einstein's equations.

The assumption $k_{13}=0$ (i.e., $g_{\alpha\beta}k_{(1)}^{\alpha}k_{(3)}^{\beta}=0$) alone does not lead to any immediate progress in solving the Einstein equations.

The Lanczos solution was originally derived in Ref. 1 (an English translation, Ref. 19, is now available), and rediscovered in Ref. 20. Its limit of zero cosmological constant was rediscovered in Ref. 21 as the cylindrically symmetric subcase of a family of stationary axially symmetric solutions. Geometrical and physical properties of the Lanczos solution were discussed in Ref. 1, and, in a more modern language, also in Ref. 22 (the latter only for the case $\Lambda=0$).

Coordinate transforms of the Gödel solution were published as new solutions in Refs. 23 and 24 (concerning Ref. 23 see also Ref. 25).

A metric form that is a modest generalization of the Gödel solution (it has two unknown functions of one variable in place of Gödel's e^{x^1} and e^{2x^1}) came to be known as ''Gödel-type metric'' and became the subject of a rather large number of papers; the activity seems to have started with Ref. 26, one of the most recent appearances of it is Ref. 27. However, it was proven already in Ref. 28 that the only perfect fluid solution with this metric is the Gödel solution itself; indeed, all other ''Gödel-type solutions'' have various nonperfect fluid sources, and therefore they do not show up in the scheme considered here.

As mentioned in Sec. I, several authors considered rotating charged dust solutions under the additional assumption that the electromagnetic field $F_{\mu\nu}$ exerts no force on the charged dust particles, i.e., that $F_{\mu\nu}u^{\nu}=0$. These solutions were all derived with another, rather natural assumption: that all charges are attached to dust particles so that no currents are present apart from the one created by the dust flow. Those solutions are found in Refs. 29–36. The one in Ref. 29 has only two-dimensional symmetry, so it could not come up in this investigation. The remaining ones are stationary and cylindrically symmetric and would have shown up here, had we allowed charges and electromagnetic fields in the source. They have the following properties.

The one from Ref. 30 becomes a vacuum solution in the limit $F_{\mu\nu}=0$.

The one from Ref. 31 does not allow this limit at all.

The limit $F_{\mu\nu}=0$ of the solution from Ref. 32 is the Minkowski metric.

The Som–Raychaudhuri solution[33] reproduces the $\Lambda=0$ subcase of the Lanczos solution when $F_{\mu\nu}=0$.

The first of the six solutions by Banerjee and Banerji[34] reduces to the Gödel solution when $F_{\mu\nu}=0$. The other five behave as follows: 2 and 6 become vacuum solutions when $F_{\mu\nu}=0$, no. 5 becomes the Minkowski spacetime, no. 3 does not allow this limit at all, and no. 4 has has a two-dimensional symmetry group.

Both solutions by Mitskiévič and Tsalakou[35] are generalizations of the Gödel solution; the first one of them is in addition a generalization of the full ($\Lambda\neq 0$) Lanczos solution (In fact, the second solution has a nonzero pressure gradient that remains nonzero even after the limit $F_{\mu\nu}=0$ is taken. Another limiting transition, given in the paper, reduces the solution to Gödel's.) In the limit $\Lambda=0$, the first solution reduces to the one by Som and Raychaudhuri.[33]

The two solutions from Ref. 36 are coordinate transforms of those from Ref. 35.

Three other generalizations of the Gödel solution exist in the literature that have zero acceleration. Two were provided by Raval and Vaidya;[37] the first of them is stationary, the second expanding, both have anisotropic pressure. The third is the solution by Rebouças[38] in which the source is a free electromagnetic field (see also Ref. 39). The metric of the Rebouças solution is the same as that in the first Banerjee–Banerji solution. This coincidence was explained by Raychaudhuri and Guha Thakurta:[40] The two electromagnetic fields (one generated by a current, the other source-free) are related by a point-dependent duality rotation.

IX. ANOTHER INVARIANT DEFINITION OF THE GÖDEL AND OZSVATH CLASS III SOLUTIONS

Assumptions about invariant properties of the velocity field of matter usually lead to progress in solving the Einstein equations; the most impressive example were the shear-free normal models of Barnes,[41] where a large class of solutions resulted from the assumptions of zero shear and zero rotation in a perfect fluid source. Inspired by this, one can try to make assumptions about other vector fields characterizing fluid sources, e.g., the rotation. Indeed, it turns out that the assumption

$$w_{\alpha;\beta}=0, \tag{9.1}$$

i.e., the rotation field being covariantly constant, together with the assumption of geodesic motion of a perfect fluid source, leads uniquely to two solutions of Einstein's equations. However, both of them were obtained before by other methods. One is the Ozsvath class III metric,[11] originally identified as one of the solutions that are homogeneous in four dimensions; the other is the Gödel solution[2] which is the shear-free limit of the Ozsvath solution.

From (9.1) and from the Ricci identity $2w_{\alpha;[\beta\gamma]}=R^{\rho}{}_{\alpha\beta\gamma}w_{\rho}$ one obtains for the Ricci tensor,

$$R^{\rho}{}_{\gamma}w_{\rho}=0, \tag{9.2}$$

and then from the Einstein equations for a perfect fluid,

$$G_{\alpha\beta}+\Lambda g_{\alpha\beta}=\kappa[(\epsilon+p)u_{\alpha}u_{\beta}-pg_{\alpha\beta}], \quad \kappa=8\pi G/c^4, \tag{9.3}$$

and from $u^{\alpha}w_{\alpha}=0$ one obtains

$$\Lambda=\tfrac{1}{2}\kappa(\epsilon-p). \tag{9.4}$$

In the case $\Lambda=0$, this is the well-known ''stiff perfect fluid.'' Equation (9.4) is a necessary condition for (9.1) when the source is a perfect fluid.

As stated at the end of Sec. III, if the perfect fluid moves geodesically with rotation, then necessarily $p=\text{const}$. Equation (9.4) implies then that $\epsilon=\text{const}$, i.e., a geodesically moving and rotating perfect fluid whose rotation vector is covariantly constant must have constant matter density. Since $\Lambda=0$ may be assumed with no loss of generality (this leads only to redefining p), we shall assume this from now on. Then $\epsilon=p$ and $\epsilon+p=2p$ is a conserved quantity. Hence, we may assume

$$n=\epsilon+p=2p=\text{const} \tag{9.5}$$

in all formulae. In particular, (3.20) implies then that

$$g=\det(g_{\alpha\beta})=-(2p)^{-2}. \tag{9.6}$$

Using (9.6) and (3.23) in (9.1) we obtain, in the Plebański coordinates,

$$\tfrac{1}{2}n(g_{\alpha3,\beta}-g_{\beta3,\alpha}+g_{\alpha\beta,3})=0. \tag{9.7}$$

After a simple algebraic manipulation this set of equations yields the following result:

$$g_{33}=-A^2=\text{const}, \quad g_{\alpha\beta,z}=0, \quad g_{13,t}=g_{23,t}=0,$$
$$g_{23,x}-g_{13,y}=0. \tag{9.8}$$

The second equation in (9.8) means that w^α is a Killing vector, as should be expected from (9.1), (9.6), and (3.23). Equations (9.8) imply that g_{13} and g_{23} depend only on x and y, and that there exists a function $\mathscr{F}(x,y)$ such that

$$g_{23}=\mathscr{F}_{,y}, \quad g_{13}=\mathscr{F}_{,x}. \tag{9.9}$$

Since we assumed that rotation is nonzero, we know that $g_{33}=-g_{\alpha\beta}w^\alpha w^\beta/n^2\neq 0$, and so we are allowed to carry out the coordinate transformation:

$$z=z'-\mathscr{F}/g_{33}, \tag{9.10}$$

that, in virtue of (9.9), will lead to

$$g_{13}=g_{23}=0 \tag{9.11}$$

in the new coordinates. We have thus arrived at the metric form:

$$ds^2=(dt+ydx)^2-h(t,x,y)[dx+k(t,x,y)dy]^2-l(t,x,y)dy^2-A^2dz^2, \tag{9.12}$$

where h, k, and l are functions to be found from the Einstein equations and A is an arbitrary constant.

From now on, the allowed coordinate transformations are (5.6)–(5.7), but with $T=\text{const}$.

The components of the Einstein tensor will now be referred to the orthonormal tetrad implied by (9.12). The equation $G_{12}=0$ is integrated with the result

$$k_{,t}=K(x,y)l^{1/2}/h^{3/2}-1/h. \tag{9.13}$$

The equation $G_{22}=\kappa p$, with l eliminated by (9.6), is integrated with the result

$$h=[H^2(x,y)+K^2/(4\kappa p)]^{1/2}+H\sin[2(\kappa p)^{1/2}t+\tau(x,y)], \tag{9.14}$$

where $H(x,y)$, $k(x,y)$, and $\tau(x,y)$ are arbitrary functions. Now $G_{11}+G_{22}=2\kappa p$ imposes an additional condition on (9.13) and (9.14) that leads to $H=0$ or

$$H=(A^2-\kappa/p)^{1/2}K/(2\kappa). \tag{9.15}$$

The case $H=0$ leads to the Gödel solution (see below), so we shall consider the more general case (9.15). Then, from (9.14),

$$h=[K/(2\kappa)]\{A+(A^2-\kappa/p)^{1/2}\sin[2(\kappa p)^{1/2}t+\tau]\}. \tag{9.16}$$

With such h, Eq. (9.13) can be integrated with the result

$$k=[2A(\kappa p)^{1/2}h]^{-1}(A^2-\kappa/p)^{1/2}\cos[2(\kappa p)^{1/2}t+\tau]+L(x,y), \tag{9.17}$$

where L is a new arbitrary function. The function l is then calculated from (9.6), and an explicit solution of Einstein's equations is determined by (9.16) and (9.17).

The transformations (5.6)–(5.7) with $T=\text{const}$ can now be used to simplify the metric tensor so that $L=0$. The transformation that yields this is given in Appendix A. The still allowed coordinate transformations that preserve the property $L=0$ are given by (5.6)–(5.7) with $T=\text{const}$ and with the additional condition

$$KF_{,x}F_{,y}+[\kappa/(A^2pK)]G_{,x}G_{,y}=0. \tag{9.18}$$

With $L=0$, from the equations $G_{01}=G_{02}=0$ one obtains further

$$\tau_{,x}=2y(\kappa p)^{1/2}-A(p/\kappa)^{1/2}K_{,y}, \qquad \tau_{,y}=-(\kappa/p)^{1/2}K_{,x}/(AK^2). \tag{9.19}$$

The integrability condition of (9.19) is

$$(p/\kappa)^{1/2}(AK)_{,yy}+(\kappa p)^{1/2}[1/(AK)]_{,xx}-2(\kappa p)^{1/2}=0. \tag{9.20}$$

By the same method as was used in Ref. 4 it can now be shown that Eq. (9.20) is at the same time the integrability condition for such a coordinate transformation (5.6)–(5.7)–(9.18) after which (see Appendix A again)

$$K=(\kappa/A)y^2, \tag{9.21}$$

and then (9.19) implies

$$\tau=c=\text{const.} \tag{9.22}$$

The value of c can be set arbitrarily by transformations of t of the form $t=t'+\text{const}$. To match Ref. 11 one should choose

$$c=-\pi/2. \tag{9.23}$$

Finally, the functions h, k, and l in (9.12) are thus

$$\begin{aligned} h&=\tfrac{1}{2}\,y^2\{1+[1-\kappa/(pA^2)]^{1/2}\cos[2(\kappa p)^{1/2}t]\},\\ k&=\tfrac{1}{2}\,[(\kappa p)^{1/2}h]^{-1}[1-\kappa/(pA^2)]^{1/2}\sin[2(\kappa p)^{1/2}t],\\ l&=(4p^2A^2h)^{-1}. \end{aligned} \tag{9.24}$$

This is equivalent under a simple coordinate transformation to the Ozsvath class III solution from Ref. 11.

The velocity field $u^\alpha=\delta^\alpha_0$ for this solution has nonzero shear. The shear will vanish if and only if:

$$A^2=\kappa/p, \tag{9.25}$$

and then the Gödel solution in the form (5.11) results.

The invariant definitions of the Ozsvath class III and of the Gödel solutions given at the end of Sec. I follow from the derivation in this section.

X. CONCLUDING REMARKS

These are the main results of the paper.

1. With nonzero rotation, any Killing field, existing for a metric whose matter source inherits the symmetry, must have the form (5.5) when represented in the Plebański coordinates. When $\phi_{;\alpha}\neq 0$, the Plebański coordinates can be adapted to the Killing field so that $k^\alpha=\delta^\alpha{}_1$.

2. When two of the generators of the group are spanned on the velocity and rotation vector fields, while the third one is not, the collection of solutions with a dust source is exhausted by the following two sets.

(a) The set of Sec. VII, defined by a single differential equation (7.24), where the metric is (7.15) with $A=k_{13}=0$, F defined by (7.23) and $y(Y)$ defined by (7.21).

(b) The set of Sec. VIII, where the metric is (7.15) with $b=0$, and the metric functions are defined by an autonomous set of first-order equations (8.1)–(8.3), (8.8), and (8.10) [the integral in (8.8) and (8.10) can be calculated if the variable is changed as in (8.9)].

3. The solutions of Lanczos[1,19] and Gödel[2] are limiting cases of both sets; their invariant definitions are given at the end of Sec. VII and of Sec. VIII.

4. With no symmetries pre-assumed, if the source is a rotating geodesic perfect fluid whose rotation vector field is covariantly constant, then the solution of the Einstein equations is the homogeneous (in four dimensions) Ozsvath class III solution.[11] If shear is zero in addition, then the Gödel solution (Ref. 2; see also Ref. 4) results.

Note the modification that the results 2(a) and 2(b) introduce in theorem 3.1 of King and Ellis.[42] Those authors considered spatially homogeneous models in which the velocity field of matter was tilted (i.e., was not orthogonal) with respect to the hypersurfaces of homogeneity. Theorem 3.1 says, among other things, that there are no tilted models of type I and that tilted models of type II have zero vorticity. Evidently, this does not apply to the case where the hypersurfaces of homogeneity are timelike. The solutions of Sec. VII are of Bianchi type II, they are ''tilted'' (because the velocity field is tangent to the hypersurfaces of homogeneity), yet rotation is not zero. The solutions of Sec. VIII are tilted in the same sense, yet they are of Bianchi type I.

Other solutions that have been published earlier will be mentioned where appropriate in papers 2 and 3. A general overview of literature on related subjects will be included in paper 3.

ACKNOWLEDGMENT

The algebraic calculations for this paper were done with the help of the program Ortocartan.[43,44]

APPENDIX A: THE TRANSFORMATION TO $L=0$ IN (9.17)

A transformation of the class (5.6)–(5.7) with $T=\text{const}$ changes the functions h and k to such ones that can be cast in the form (9.16) and (9.17), respectively, with the new functions K', τ', and L' expressed through the old ones as follows:

$$K' = K(F_{,x'} + LG_{,x'})^2 + \kappa G_{,x'}^2/(A^2 pK), \tag{A1}$$

$$\tau' = \tau - 2(\kappa p)^{1/2} S + U, \tag{A2}$$

$$L' = K'^{-1}[KF_{,x'}F_{,y'} + 2KLF_{,y'}G_{,x'} + KL + KL^2 G_{,x'}G_{,y'} + \kappa G_{,x'}G_{,y'}/(A^2 pK)], \tag{A3}$$

where S in (A2) is the function from (5.6)–(5.7) and U is determined by

$$\cot U = 2A(\kappa p)^{1/2}[2G_{,x'}(F_{,x'} + LG_{,x'})]^{-1}[-K(F_{,x'} + LG_{,x'})^2/(2\kappa) + G_{,x'}^2/(2A^2 pK)]. \tag{A4}$$

Note that we are applying here (5.6)–(5.7) in reverse, i.e., with the roles of x^α and x'^α interchanged. The functions of the inverse transformation, denoted again by S, F, G, and T, still obey (5.7). For consistency of all the formulae it is convenient to choose U from the segment $(\pi, 2\pi)$. Then, the limiting cases $G_{,x'}=0$ and $F_{,x'}+LG_{,x'}=0$ are included in (A4) as the limits $U=2\pi$ and $U=\pi$, respectively (these limiting cases occur when $L_{,xx}=0$ in the original coordinates).

From (A3), the equation $L'=0$ turns out to be consistent with (5.6)–(5.7). In order to see this, one can solve (A3) and (5.7) for $F_{,x'}$ and $F_{,y'}$ and then impose the integrability condition $F_{,x'y'} - F_{,y'x'} = 0$. What comes out is a well-defined (though highly nonlinear) partial differential equation of second order for G whose coefficients depend only on K and L.

The transformation preserving the property $L=0$ are (5.6)–(5.7) with (9.18), the latter easily follows from (A3). Equation (A1) with $L=0$ then shows how K is changed by such a transformation; this is useful in showing that coordinates exist in which $K_{,x}=0$ [see the remark after (9.20)]. The proof is identical as in Appendix C to Ref. 4. Note that the conclusion in Ref. 4 is weaker than it could be: the $K(x)$ [$v(t)$ in Ref. 4] is determined up to an additive constant C. Hence, by a transformation of the form $y=y'+\text{const}$ and by an appropriate choice of C one can remove the linear and the constant terms in K (resp., v) so that $K\propto y^2$ (resp., $v\propto t^2$ in Ref. 4).

[1] K. Lanczos, Z. Phys. **21**, 73 (1924).
[2] K. Gödel, Rev. Mod. Phys. **21**, 447 (1949).
[3] J. Plebański, *Lectures on Non-Linear Electrodynamics* (Nordita, Copenhagen, 1970), pp. 107–115 and 130–141.
[4] A. Krasiński, Acta Phys. Pol. B **5**, 411 (1974).
[5] A. Krasiński, Acta Phys. Pol. B **6**, 223 (1975).
[6] A. Krasiński, Acta Phys. Pol. B **6**, 239 (1975) (available in extended form as a preprint).
[7] A. Krasiński, J. Math. Phys. **16**, 125 (1975).
[8] A. Krasiński, Rep. Math. Phys. **14**, 225 (1978).
[9] C. B. Collins, Can. J. Phys. **64**, 191 (1986).
[10] H. Stephani, Class. Quantum Grav. **4**, 125 (1987).
[11] I. Ozsvath, J. Math. Phys. **11**, 2871 (1970).
[12] S. Sternberg, *Lectures on Differential Geometry* (Prentice–Hall, Englewood Cliffs, NJ, 1964), p. 141.
[13] G. F. R. Ellis, in ''General relativity and cosmology,'' *Proceedings of the International School of Physics ''Enrico Fermi*,'' Course 47, edited by R. K. Sachs (Academic, New York, 1971), p. 104.
[14] J. Ehlers, Abhandl. Math. Naturw. Kl. Akad. Wiss. Lit. Mainz **11**, 791 (1961); English translation in Gen. Relativ. Gravit. **25**, 1225 (1993).
[15] G. F. R. Ellis, J. Math. Phys. **8**, 1171 (1967).
[16] E. Verdaguer, Phys. Rep. **229** No. 1, 1 (1993).
[17] O. J. Bogoyavlenskii, *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics* (Springer, Berlin, 1980).
[18] H. Stephani, *General Relativity*, 2nd ed. (Cambridge University Press, Cambridge, 1990), pp. 284–288.
[19] K. Lanczos, Gen. Rel. Grav. **29**, 363 (1997).
[20] J. P. Wright, J. Math. Phys. **6**, 103 (1965).
[21] W. J. van Stockum, Proc. R. Soc. Edinburgh **57**, 135 (1937).
[22] W. B. Bonnor, J. Phys. A **13**, 2121 (1980).
[23] C. Hoenselaers and C. V. Vishveshwara, Gen. Relativ. Gravit. **10**, 43 (1979).
[24] A. H. Khater and M. F. Mourad, Astrophys. Space Sci. **163**, 247 (1990).
[25] S. K. Chakraborty, Gen. Relativ. Gravit. **12**, 925 (1980).
[26] M. Novello and M. J. Rebouças, Phys. Rev. D **19**, 2850 (1979).
[27] R. X. Saibatalov, Gen. Relativ. Gravit. **27**, 697 (1995).
[28] F. Bampi and C. Zordan, Gen. Relativ. Gravit. **9**, 393 (1978).
[29] J. N. Islam, Proc. R. Soc. London, Ser. A **353**, 523 (1977).
[30] J. N. Islam, Proc. R. Soc. London, Ser. A **385**, 189 (1983).
[31] P. Wils and N. van den Bergh, Proc. R. Soc. London, Ser. A **394**, 437 (1984).
[32] A. Georgiou, Nuovo Cimento B **108**, 69 (1993).
[33] M. M. Som and A. K. Raychaudhuri, Proc. R. Soc. London, Ser. A **304**, 81 (1968).
[34] A. Banerjee and S. Banerji, J. Phys. A **1**, 188 (1968).
[35] N. V. Mitskiévič and G. A. Tsalakou, Class. Quantum Grav. **8**, 209 (1991).
[36] A. M. Upornikov, Class. Quantum Grav. **11**, 2085 (1994).
[37] H. M. Raval and P. C. Vaidya, Ann. Inst. Henri Poincaré, Sect. A **4**, 21 (1966).
[38] M. J. Rebouças, Phys. Lett. A **70**, 161 (1979).
[39] M. J. Rebouças and J. Tiomno, Phys. Rev. D **28**, 1251 (1983).
[40] A. K. Raychaudhuri and S. N. Guha Thakurta, Phys. Rev. D **22**, 802 (1980).
[41] A. Barnes, Gen. Relativ. Gravit. **4**, 105 (1973).
[42] A. R. King and G. F. R. Ellis, Commun. Math. Phys. **31**, 209 (1973).
[43] A. Krasiński, Gen. Relativ. Gravit. **25**, 165 (1993).
[44] A. Krasiński and M. Perkowski, Gen. Relativ. Gravit. **13**, 67 (1981).