SPHERICALLY SYMMETRIC SHEARFREE UNIVERSES WITH A BAROTROPIC EQUATION OF STATE

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Wyman (1946) found the unique spherically symmetric inhomogeneous solution of Einstein's equations where the source is a shearfree expanding barotropic perfect fluid. Two independent proofs of this statement are reconciled: a missing detail in Wyman's proof is reinserted, and the additional assumption Srivastava and Prasad (1983) is shown to be automatically fulfilled for a perfect fluid. The uniqueness of Wyman's solution is shown to imply that the barotropic equation of state is a strong and rather artificial constraint on cosmological models.

1. THE PROBLEM.

The problem of solving the Einstein field equations for a spherically symmetric shear-free expanding barotropic perfect fluid was posed for the first time by Wyman[1] in 1946 (though in the language of then the words "shear" and "expansion" were not used). He derived the field equations for the problem, then presented one solution to them (in fact two, but the second one is a coordinate transform of the first) and stated that it was the only solution apart from the Friedman-Lemaitre-Robertson-Walker metrics (FLRW - the term also not in use by then). This statement looks suspicious because Wyman apparently chose certain values of free parameters in his equations. However, the statement is correct. After arriving at his set of equations (let us call it SE) Wyman did not warn the reader that the condition $p = p(\rho)$ was not fully used up yet, although his previous reasoning suggests that it was. Consequently, each solution of SE had to be checked for consistency with $p = p(\rho)$, and in this way Wyman established the uniqueness of his metric. In sec. 2 of this paper, the condition $p = p(\rho)$ is reformulated as an additional equation for the metric functions which then uniquely selects Wyman's metric from all possible solutions of SE.

Srivastava and Prasad[2] (abbreviated S-P) repeated Wyman's calculations in a different coordinate system and confirmed his result. However, their proof contains a simplifying assumption of mysterious origin that arouses suspicions about the generality of the conclusion. It turns out that S- P omitted an equation. Their assumption follows from the equation they omitted and so is fulfilled automatically (sec. 3).

The uniqueness of the Wyman solution was confirmed also by Collins and Wainwright[3] who noticed that the simplifying assumption of S-P is superfluous for the proof, but did not notice that it is always fulfilled.

Sec. 4 discusses the papers by McVittie[4] and Taub[5] who hit upon the Wyman's problem independently, apparently unaware of Ref. 1. McVittie's solutions include Wyman's as a subcase, but do not include the FLRW models. Taub rederived Wyman's equations without discussing any solutions.

Finally, it is argued in sec. 5 that the barotropic equation of state turns out, in the light of Wyman's result, to be a very stong and rather artificial constraint on cosmological models. Wyman's solution in shown to reduce in the limitting case of spatial homogeneity to the de Sitter solution. Hence, the equation of state $p = p(\rho)$ selects from among spherically symmetric shearfree expanding perfect fluid solutions a rather small subset, leaving the FLRW models without inhomogeneous parent solutions.

2. THE FULL SET OF WYMAN'S EQUATIONS.

We shall quote here only the main points of Wyman's reasoning without repeating the proofs, and reinsert the information omitted.

Wyman began with the metric:

$$ds^2 = \mathrm{e}^\nu\, dt^2 - \mathrm{e}^\mu (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{2.1}$$

where $\nu(t,r)$ and $\mu(t,r)$ are functions to be determined from the field equations. He assumed that the source is a perfect fluid with the barotropic equation of state, $p = p(\rho)$, while the coordinates of (2.1) are comoving, so the velocity field is $u^\alpha \underset{*}{=} \mathrm{e}^{-\nu/2}\, \delta^\alpha_0$. From the field equations he concluded that:

$$\mathrm{e}^\nu = \mathrm{e}^{\Phi(t)}\, \dot{\mu}^2, \tag{2.2}$$

where $\dot{\mu} = \partial\mu/\partial t$, while $\Phi(t)$ is an arbitrary function, and that e^μ must obey the equation:

$$\mathrm{e}^{\mu/2}(\mu" - \mu'^2/2 - \mu'/r) = \psi(r), \tag{2.3}$$

where $\mu' = \partial\mu/\partial r$ and $\psi(r)$ is another arbitrary function. The case $\dot{\mu} = 0$ implies a static solution and is not considered. Let us note that with the substitutions $\mathrm{e}^\mu = 1/F^2$ and $u = r^2$, eq. (2.3) changes to:

$$F_{,uu}/F^2 = f_{KQ}(u), \tag{2.4}$$

where $f_{KQ}(u) = -\psi(r(u))/(8u)$. In the form (2.4), and without any assumption about the equation of state, the field equations for (2.1) were first thoroughly investigated by

Kustaanheimo and Qvist[6] (see also Ref. 7). Eq. (2.3) follows from the requirement that the energy-momentum tensor has a triply degenerated eigenvalue (i.e. that the pressure is isotropic in the sense of the Pascal law), so it is often called the **isotropy condition**.

With (2.2) and (2.3) the field equations just define pressure p and matter-density ρ. The requirement that the equation of state is barotropic imposes further limitations. From the equations of motion, $T^{\mu\nu};_{\nu} = 0$, it follows that:

$$p' = -(\rho + p)\dot{\mu}'/\dot{\mu}, \tag{2.5}$$

$$\dot{\rho} = -\frac{3}{2}(\rho + p)\dot{\mu}. \tag{2.6}$$

From here it follows:

$$(\rho + p)\dot{\rho}' = \dot{\rho}\rho'. \tag{2.7}$$

Since $p = p(\rho)$ by assumption, the function $q(\rho)$ may be defined:

$$\frac{dq}{d\rho} = (\rho + p)^{-1}, \tag{2.8}$$

and then it follows further that

$$\rho = \rho(v), \tag{2.9}$$

where

$$v = t + k(r), \tag{2.10}$$

$k(r)$ being an arbitrary function. Actually, (2.7) and (2.8) imply $v = \alpha(t) + k(r)$ where α is another arbitrary function, but since $\alpha,_t \neq 0$ by assumption (otherwise a static solution results), $\alpha(t)$ may be chosen as the new t-coordinate, with (2.10) resulting. Then (2.6) may be integrated:

$$\mu = -\frac{2}{3}q(\rho) + f(r), \tag{2.11}$$

where $f(r)$ is one more arbitrary function. Wyman then introduces a set of new symbols:

$$x \stackrel{\rm def}{=} r^2/2, \qquad \psi((2x)^{1/2}) \stackrel{\rm def}{=} 2xh(x), \qquad k((2x)^{1/2}) \stackrel{\rm def}{=} y(x),$$
$$f((2x)^{1/2}) \stackrel{\rm def}{=} z(x), \qquad q(\rho(v)) \stackrel{\rm def}{=} -\frac{3}{2}F(v). \tag{2.12}$$

With these, eq. (2.11) substituted into (2.3) implies:

$$\mathrm{e}^{(F+z)/2}[(F,_{vv} - \frac{1}{2}F,_v^2)y,_x^2 + F,_v(y,_{xx} - y,_x z,_x) + z,_{xx} - \frac{1}{2}z,_x^2] = h(x). \tag{2.13}$$

Since $\dot{v} \neq 0$, the variables v and x can be treated as independent. This implies certain conditions on F, y, z and h. Wyman concludes that the conditions are:

$$y,_{xx} - y,_x z,_x = ay,_x^2, \tag{2.14}$$

$$z,_{xx} - \frac{1}{2}z,_x^2 = by,_x^2, \tag{2.15}$$

$$\mathrm{e}^{F/2}(F,_{vv} - \frac{1}{2}F,_v^2 + aF,_v + b) = c, \tag{2.16}$$

$$h = c\,\mathrm{e}^{z/2}\, y,_x^2, \tag{2.17}$$

where a, b and c are arbitrary constants. The deduction of (2.14) - (2.17) from (2.13) is more subtle than Ref. 1 allows to guess, but it is true that (2.14) - (2.17) follow from (2.13) apart from special cases which lead either to static solutions or to FLRW models.

Eqs. (2.14) - (2.17) allow a multitude of solutions for y, z and h (all of them elementary), and it is surprising to read in Wyman's paper: "The author has carried through a complete investigation of every case possible and only two new solutions exist" (one of them follows when $a = b = y,_{xx} = 0$, the other when $a = b = (1/y),_{xx} = 0$ and is related to the first by the coordinate transformation $r = 1/r'$). The fact is that eqs. (2.9) - (2.10) are only necessary, but not sufficient conditions for (2.7) - (2.8) to hold. Therefore in deriving

(2.14) - (2.17) the condition $p = p(\rho)$ was not fully used up and every solution of (2.14) - (2.17) must be checked for consistency with $p = p(\rho)$. This procedure is illustrated by Wyman's case 2 in his sec. 3, but the necessity of the check is not verbalized in the paper. The check runs as follows. Eqs. (2.7) - (2.8) imply

$$\int \mathrm{e}^{-q(\rho)}\, d\rho = v, \tag{2.18}$$

from which it follows, on differentiating by r that:

$$\rho_{,r} = \mathrm{e}^{q}\, k_{,r} = \mathrm{e}^{-\frac{3}{2}F}\, k_{,r}\,. \tag{2.19}$$

Now, a formula for ρ results from the field equations (Wyman's eq. (2.8)):

$$\kappa\rho = -\,\mathrm{e}^{-\mu}(\mu" + \mu'^2/4 + 2\mu'/r) + \frac{3}{4}\,\mathrm{e}^{-\Phi}, \tag{2.20}$$

($\kappa \stackrel{\mathrm{def}}{=} 8\pi G/c^4$), and from here, using (2.11), (2.12) and (2.3) it follows that:

$$\kappa\rho_{,r} = -(5rh + r^3 h_{,x})\,\mathrm{e}^{-3\mu/2}\,. \tag{2.21}$$

Comparing (2.19) and (2.21), and using (2.12) again we obtain:

$$5h + 2xh_{,x} = -\kappa\,\mathrm{e}^{\frac{3}{2}z}\, y_{,x}\,. \tag{2.22}$$

Eq. (2.22) makes full use of (2.7) - (2.8), and it selects uniquely the two Wyman's metrics from the set of inhomogeneous ($y_{,x} \neq 0$) solutions of (2.14) - (2.17). Hence, the full set of equations to be considered consists of (2.14) - (2.17) and (2.22).

Wyman took note of the fact that with $\psi = 0$ eq. (2.3) is fulfilled by $\mathrm{e}^{\mu} = [A(t)r^2 + B(t)]^{-2}$ where A and B are arbitrary functions. This is the spherically symmetric subcase of the Stephani solution[8−10]. Had the author taken it seriously, he would have been its first discoverer. However, he dismissed it instantly observing that it leads to $\rho = \rho(t)$ what is possible with $p = p(\rho)$ only if also $p = p(t)$, i. e. only for the FLRW metrics. The credit for this discovery must thus go to Kustaanheimo and Qvist[6].

3. THE FLAW IN THE ARGUMENT OF SRIVASTAVA AND PRASAD.

They begin with the metric:

$$ds^2 = \mathrm{e}^{\nu}\, dt^2 - \mathrm{e}^{\mu}(dr^2 + d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where $\nu = \nu(t, r), \mu = \mu(t, r)$. Note that the coordinates and functions of (3.1) are not identical with those of (2.1). Marking the quantities from (3.1) by the subscript SP, the transformation to (2.1) is:

$$r_{SP} = \ln r, \qquad \mu_{SP} = \mu + 2\ln r. \tag{3.2}$$

There are more clashes of notation between Refs. 1 and 2. Therefore we shall stick to each paper's original notation, and only point out the equivalence between formulae. The following dictionary relates the various symbols of Wyman and S-P:

S-P	r	e^{μ}	x	σ	X	B
Wyman	$\ln r$	$r^2 e^{\mu}$	e^{v}	e^{q}	$e^{-F/2}$	e^{y}

(3.3)

S-P	C	A	α	β	γ
Wyman	$(2x)^{1/2}e^{z/2}$	$2^{3/2}x^{5/2}h$	$2/c$	$2(a+1)/c$	$-b/c$

S-P consider the field equations together with the equations of motion in the form:

$$2p' = -(\rho + p)\nu', \tag{3.4}$$

$$2\dot{\rho} = -3(\rho + p)\dot{\mu}, \tag{3.5}$$

$$m' = \frac{1}{4}\kappa\rho e^{3\mu/2}\mu', \tag{3.6}$$

$$\dot{m} = -\frac{1}{4}\kappa p e^{3\mu/2}\dot{\mu}, \tag{3.7}$$

where

$$8m \stackrel{\mathrm{def}}{=} \dot{\mu}^2 e^{3\mu/2-\nu} - \mu'^2 e^{\mu/2} + 4e^{\mu/2}. \tag{3.8}$$

They conclude from (3.6) - (3.8) that

$$3m = \frac{1}{2}\kappa\rho e^{3\mu/2} + A(r) + N(t), \tag{3.9}$$

where $A(r)$ and $N(t)$ are arbitrary functions. Then they assume $N(t) = \text{const}$ (so $N = 0$ by redefining A), arguing that this is necessary for the metric to be regular at $r = 0$. With $N = 0$ they find that the assumption $p = p(\rho)$ and eqs. (3.6) - (3.9) imply:

$$(e^{-\mu/2})'' = e^{-\mu/2} - Ae^{-\mu}. \tag{3.10}$$

Eq. (3.10) is precisely the isotropy condition (2.3) what may be verified with use of (3.2). Without the assumption $N = 0$ eq. (3.10) would contain the additional term $(-Ne^{-\mu})$ on the right-hand side. Then, however, the isotropy condition would imply $N = 0$, otherwise the source would not be a perfect fluid. Hence $N = 0$ need not be assumed, and eq. (3.10) applies generally. S-P found the assumption necessary because they left out the isotropy condition from the set (3.5) - (3.8). In fact, one more equation is missing from the set, it is $2\dot{\mu}' - \dot{\mu}\nu' = 0$ (only then is the Einstein tensor diagonal), but S-P recovered this one as the integrability condition of (3.4) - (3.5).

S-P further derive a set of equations that is equivalent to (2.14) - (2.17) (this may be verified with use of (3.3)), and they do not neglect (2.22). In this way they conclude that Wyman's solution is indeed unique.

4. TWO OTHER RELATED STUDIES.

The papers of McVittie[4] and Taub[5] discussed in this section touched upon the Wyman problem only marginally, but they must be mentioned here in order to avoid a possible confusion.

McVittie assumed the metric to be:

$$ds^2 = y^2 dt^2 - R_0^2 S^2 e^\eta [dr^2 + f^2(d\theta^2 + \sin^2\theta d\phi^2)], \qquad (4.1)$$

where $S(t)$ and $f(r)$ are arbitrary functions, η is **assumed** to be a one-argument function of the variable z defined by:

$$e^z \stackrel{\mathrm{def}}{=} Q(r)/S(t), \qquad (4.2)$$

$Q(r)$ is another arbitrary function, and:

$$y = 1 - \eta,_z /2. \qquad (4.3)$$

The last relation follows from the (0 1) field equation if the coordinates of (4.1) are co-moving and the source is a perfect fluid.

The equations (4.1) - (4.3) follow from the field equations and the equations of motion if it is assumed that the perfect fluid source obeys a barotropic equation of state. This was demonstrated by Taub[5] (and, in fact, by Wyman[1]). However, the converse is not true: if (4.1) - (4.3) are assumed, then $p = p(\rho)$ need not hold (see also Ref. 5). Discarding $p = p(\rho)$ for (4.1) - (4.3) is equivalent, in Wyman's notation, to dropping eq. (2.22) while retaining (2.14) - (2.17). With (2.22) dropped, the set (2.14) - (2.15) admits several solutions which can be **all** obtained by elementary though laborious calculations. Then h can be calculated from (2.17) for each solution of (2.14) - (2.15). Eq. (2.16) is independent of the others and is the only one that leads to transcendental functions.

Although the equations derived by McVittie (equivalent to our (2.14) - (2.17) with (2.22) omitted) are fulfilled by the Wyman and the FLRW metrics, the specific examples presented in Ref. 4 include only the Wyman solution as a subcase. The FLRW models which would result with Q = const are explicitly excluded from consideration. In McVittie's notation, the Wyman model results when $f = r, a = 1$ and $b = 0$.

Taub[5] rederived the set equivalent to our eqs. (2.14) - (2.17) in a notation that was quite closely followed by S- P[2] and in our sec. 3, but omitted the equivalent of eq. (2.22). However, he did not attempt to solve the equations (Ref. 5 is mostly devoted to the Kustaanheimo-Qvist problem[6]).

5. THE BAROTROPIC EQUATION OF STATE AS A SELECTING AGENT IN THE SET OF SOLUTIONS.

The Wyman's solution is given by (2.1) with:

$$e^{\mu} = c^2/(144w^2), \tag{5.1}$$

$$e^{\nu} = w,_v^2/[2(\alpha C_2 C_3 t + \beta)w^2], \tag{5.2}$$

where

$$v \stackrel{\mathrm{def}}{=} t + \frac{1}{2}C_2 r^2, \tag{5.3}$$

C_2, C_3, c and β are arbitrary constants, $\alpha = 144/c^2, w$ is a special case of the Weierstrass elliptic function[11] defined by the equation:

$$w,_v^2 = 4w^3 - C_3, \tag{5.4}$$

and the matter density and pressure are:

$$\kappa\rho = 6\alpha C_2(ww,_v + C_3 v) + K, \tag{5.5}$$

$$\kappa p = -\alpha C_2(ww,_v + 6C_3 v - 5C_3 w/w,_v) - K, \tag{5.6}$$

where $K =$ const (there is a misprint in Ref. 1, the factor C_2 in (5.6) is left out). Since the FLRW models fulfil all the initial assumptions of Wyman's paper (i.e. are spherically symmetric, shearfree, expanding and admit a barotropic equation of state), one should expect to find them among the solutions. They should result when ρ and p become spatially homogeneous, i.e. cease to depend on r. This happens when either $\rho,_v = p,_v = 0$ or $v,_r = 0$. The first possibility leads to a static solution. In the second case, $C_2 = 0$. Then, however, $\rho = -p =$ const, and the de Sitter metric results, the source degenerating into the cosmological term. This means that within the family of spherically symmetric shearfree expanding perfect fluid spacetimes the barotropic equation of state selects a "fiber" over the de Sitter spacetime, leaving the FLRW models as isolated points without any inhomogeneous parent solutions. Note that without the assumption $p = p(\rho)$, the expanding perfect fluid solutions for the metric (2.1) are determined by (2.3) and (2.2), i.e. any function $\psi(r)$ generates a solution of (2.3) which then depends on two arbitrary functions of t, so indeed $p = p(\rho)$ throws out a lot (see also Ref. 7 and references therein). All the FLRW models result from (2.3) when $\psi = 0$ and $\mu = \ln[R^2(t)/(1 + \frac{1}{4}kr^2)^2]$.

The barotropic equation of state turns out to be even more restrictive when applied to plane and hyperbolically symmetric solutions (with other assumptions as by Wyman). They were considered, along with the spherically symmetric ones, by Collins and Wainwright[3] (the solutions were originally derived by Barnes[12], their relation to the FLRW models was discussed in Ref. 7). In the plane symmetric case, the condition $p = p(\rho)$ also selects a generalization of the de Sitter metric only, and necessarily brings in an additional symmetry. In the hyperbolic case, it kills off all inhomogeneous spacetimes, letting only the FLRW models survive (see Ref. 3).

This shows that the requirement $p = p(\rho)$ is a very strong constraint on cosmological models. It seems unnatural in inhomogeneous models because it forces the entropy per particle to be a universal constant (see e.g. Ref. 13) where most other scalar quantities vary in space and time. Thus the habit of calling $p = p(\rho)$ **the** equation of state and claiming that with $dp \wedge d\rho \neq 0$ no equation of state exists (or, less extremely, no "reasonable" one) is itself not necessarily reasonable (see also Ref.14).

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