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A NOTE ON THE UNIQUENESS OF THE WYMAN SOLUTION

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It is pointed out that the barotropic equation of state splits the family of spherically symmetric shearfree expanding perfect fluid solutions of Einstein's equations into two distinct families: the Wyman solution and the Robertson–Walker (R–W) solutions. The latter ones are not contained in the former as a limiting case. Based on this fact, it is argued that the barotropic equation of state is unnatural in inhomogeneous cosmological models.

Wyman [1] found a spherically symmetric (inhomogeneous) solution of Einstein's equations, where the source is a shearfree expanding barotropic perfect fluid. The Robertson–Walker (R–W) models are not, however, contained in his solution in the limiting case of spatial homogeneity, even though they fulfil all the same assumptions. This may possibly raise doubts about the correctness of Wyman's statement that his solution is unique, especially in view of a certain gap in his argument. The gap is the following: having apparently solved all the equations, he substituted the solutions back into the equations and found that only some of the solutions really fulfil the equations. This is in fact only a lapse of presentation, as shown below: the statement is correct, and the proof very nearly so. The uniqueness of the Wyman solution demonstrates that the barotropic equation of state is a strong and rather artificial constraint on cosmological models.

The gap in Wyman's presentation is in his eqs. (2.16)–(2.17): they are a necessary, but not a sufficient condition for the barotropic equation of state to hold. This is the reason why every solution of his eqs. (3.7)–(3.9) had to be checked again for consistency with $\mu = \mu(p)$. Wyman's eqs. (3.7)–(3.9) are equivalent to Mc Vittie's [2] eqs. (A.6)–(A.8) and allow a multitude of non-barotropic solutions.

An independent proof of uniqueness of the Wyman solution was published by Srivastava and Prasad [3]. In their proof, an additional assumption is made (that the metric is "regular" at the centre of symmetry). The question then arises, why Wyman's proof worked without that assumption. On closer inspection it turns out that Srivastava and Prasad did not make use of the isotropy of pressure (i.e. of the fact that three eigenvalues of the energy-momentum tensor must be equal for a perfect fluid). With the isotropy assumed, their regularity condition follows automatically.

The uniqueness of the Wyman solution was also proven by Collins and Wainwright [4]. However, their method was still different and relied heavily on a rather complicated derivation of the shearfree normal perfect fluid solutions by Barnes [5]. Therefore it did not reveal the gaps in the other two proofs, and would be much more difficult to verify.

Having thus removed (hopefully) any possible doubts about the uniqueness of the Wyman solution, let us observe the following. The pressure p and the energy-density μ in the model both depend only on the variable $v = t+\frac{1}{2}C_2 r^2$, where C_2 is an arbitrary constant and (t, r) are the time and the radial coordinate (the velocity field has only the t-component). Hence, μ and p will become spatially homogeneous when either $\mu_{,v} = p_{,v} = 0$ or $C_2 = 0$. In the first case, a static model results. In the second case, the result is $\mu = -p = \text{const}$, i.e. the Wyman solution degenerates then to a vacuum solution with the Λ-term. (This turns out to be the de Sitter model, in order to see this, a misprint in the formula for p has to be corrected, the correct formula is $\varkappa p = -\alpha C_2\ [ww_{,v}+6C_3 v-5C_3 w/w_{,v}]-K$.) This is rather surprising because the R–W models fulfil all the assumptions leading to the Wyman solution (they are spherically symmetric, shearfree, expanding and compatible with a barotropic equation of state). Therefore, they might be expected to be the homogeneous limiting cases of the Wyman solution. But they are not: the barotropic equation of state splits the family of spherically symmetric shearfree perfect fluid solutions into two distinct subsets: the Wyman solution and the R–W solutions, leaving the latter ones without an inhomogeneous parent solution. (The common subset of the two is only the de Sitter model.)

In order to see how restrictive the assumption $\mu = \mu(p)$ really is, let us observe that all the spherically symmetric shearfree nonstatic perfect fluid solutions are generated by the equation

$$R(t)\, V_{,uu}/V^2 = f(u), \tag{1}$$

where $R(t)$ and $f(u)$ are arbitrary functions, $u = r^2 = x^2+y^2+z^2$, and the metric is given by

$$D = F(t)(R/V)(V/R)_{,t} \tag{2}$$

$$ds^2 = D^2 dt^2-(R^2/V^2)(dx^2+dy^2+dz^2), \tag{3}$$

where $V = V(t, u)$ (this result was first obtained by Kustaanheimo and Qvist [6]). The pressure and energy density are then simply defined by the Einstein's equations. The general solution will then be labelled by three arbitrary functions of one variable: $f(u)$ and the two functions of t that will arise as integration constants in (1), and the R–W limit will result when $f(u) = 0$ and $V_{,t} = 0$. The barotropic equation of state forces all the three functions to be constants, and leads to a qualitatively different result when $f \neq 0$ (the Wyman solution) and when $f = 0$ (the R–W models).

The condition $\mu = \mu(p)$ is even more restrictive in the case of shearfree plane symmetric and shearfree hyperbolically symmetric solutions considered by Collins and Wainwright [4]. In the inhomogeneous plane symmetric case, the barotropic equation of state selects another generalization of the de Sitter model (with no nontrivial R–W limit) and necessarily brings in an additional symmetry. In the hyperbolically symmetric case, it kills off all inhomogeneous spacetimes, and selects just the R–W models.

With $\mu = \mu(p)$, the entropy per baryon in a perfect fluid becomes a universal constant (see Ref. 7). This is natural (indeed, necessary) in a spatially homogeneous model, but finds no natural justification in an inhomogeneous model where most physical quantities vary in space. Thus the habit of calling $\mu = \mu(p)$ the equation of state and claiming that with $dp \wedge d\mu \neq 0$ no equation of state exists (or, less extremely, no "reasonable" one) is itself not necessarily reasonable (see also Ref. 8).

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