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7

Early inhomogeneous cosmological models in Einstein's theory

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7.1 Introduction

Just as everyone knew in the 1920s that the universe was static (Ellis 1988b), so almost everyone knew for sure into late 1970s that the universe was homogeneous. Therefore those works which explored the possibility of our universe being lumpy were ignored by mainstream cosmologists. However, a few early papers contain surprisingly powerful results. Here, an attempt is made to pull them out of the shade and give them the credit they deserve.

The investigation underlying this discussion was an offspring of a review of all exact solutions of Einstein's equations which generalize the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological models. The main review is not yet completed, so it is possible that other papers will be added at a later date to the present collection. In accordance with the framework of this symposium, the only generalizations of the FLRW models which will be discussed are those found before 1970. However, it is worth knowing that many more were found later, and the main review includes so far over seventy independently obtained (though not always independent) solutions.

Each of the sections 7.2 to 7.6 contains a concise presentation of one particular class of models, namely:

Section 7.2: the model found by Lemaître (1933), known under the name Tolman–Bondi, and the considerations of Tolman (1934) and Bonnor (1956) which were based on it.
Section 7.3: the model of McVittie (1933) which is a superposition of the Schwarzschild and the FLRW solutions.
Section 7.4: the spherically symmetric shearfree barotropic solution of Wyman (1946).
Section 7.5: the spherically symmetric shearfree perfect fluid spacetimes of Kustaanheimo and Qvist (1948).
Section 7.6: the general conformally flat expanding perfect fluid solution of Stephani (1967).

In section 7.7 note is taken of a few historical curiosities of a lesser physical significance.

7.2 The Lemaître–Tolman model

The solution of Einstein's equations underlying this model was found by Lemaître (1933). It became later known as the "Tolman–Bondi model," even though Tolman (1934) referred to Lemaître, and Bondi (1947) quoted Tolman, and none of the latter authors claimed priority. The model should thus properly be called after Lemaître, but in order to avoid confusion with the FLRW models I propose to call it Lemaître–Tolman (L-T).

In a convenient notation introduced by Zeldovich and Grishchuk (1989) the metric of the model is

$$ds^2 = dt^2 - \frac{r'^2 dR^2}{1+f(R)} - r^2(t,R)\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $f(R)$ is an arbitrary function, the prime denotes $\partial/\partial R$, $r(t,R)$ is determined by the equation

$$\left(\frac{\partial r}{\partial t}\right)^2 = \frac{F(R)}{r} + f(R) + \frac{1}{3}\Lambda r^2, \tag{2}$$

$F(R)$ is another arbitrary function and Λ is the cosmological constant. The matter density in the model is

$$\varkappa\varrho = \frac{F'}{r^2 r'}, \tag{3}$$

where $\varkappa = 8\pi G/c^2$ and the pressure is identically zero. The FLRW limit is obtained when

$$r = RS(t),\; f = -kR^2,\; F = CR^3, \tag{4}$$

where $S(t)$ is the FLRW scale function, k is the FLRW curvature index ($k = \pm 1, 0$), and C is a constant. In the limit, (2) becomes the Friedmann equation (with the mass-conservation integral already incorporated).

Using this solution Tolman (1934) showed that the Einstein static universe is unstable against a perturbation different from the one considered by Eddington (1930) (where the matter density was knocked off its Einstein value and allowed to vary in time, but forced to remain constant throughout space at each given moment). The L-T model can be thought of as such a perturbation of the Einstein universe in which the matter density is allowed to vary both in space and time. If the density is initially non-uniform in space, then the evolution determined by (2) will enhance rather than suppress the inhomogeneity. This is simultaneously an instability of the FLRW models against the growth of inhomogeneities. Tolman also observed that the universe could contain several homogeneous regions, each described by a FLRW metric and having a different matter-density, with transition zones interpolating between them described by L-T metrics. In such a universe, each FLRW zone would "then behave as in some particular completely homogeneous model without reference to the behavior of other parts of the model." Tolman concluded his paper as follows:

> Hence, it would appear wise at the present stage of theoretical development, to envisage the possibility that regions of the universe beyond the range of our present telescopes might be contracting rather than expanding and contain matter with a density and stage of evolutionary development quite different from those with which we are familiar. It would also appear wise not to draw too definite conclusions from the behavior of homogeneous models as to a supposed initial state of the whole universe.

The paper contains other statements that sound surprisingly modern:

> it is evident that some preponderating tendency for inhomogeneities to disappear with time would have to be demonstrated, before such models could be used with confidence to obtain extrapolated conclusions as to the behavior of the universe in very distant regions or over exceedingly long periods of time. ["such models" here means homogeneous] ... in regions where the density starts to increase it is evident from the full form of equation (18) that reversal in the process of condensation would not occur short of arrival at a singular state involving infinite density or of the breakdown in our simplified equations. [Tolman's equation (18) expresses $[\partial^2(\log \varrho)]/[\partial t^2]$ through ϱ and the metric functions.]

Bonnor (1956) used the L-T model to discuss the evolution of a localized condensation of matter in the universe. He assumed that the condensation is a sphere of a FLRW space surrounded by an L-T transition zone which is in turn surrounded by another FLRW space with a different density than in the central region. By investigating the evolution of such condensations Bonnor found that perturbations of homogeneous distribution of matter may produce galaxies by today only if they are initially several orders of magnitude larger than purely statistical fluctuations in density. This was a problem of much debate in the 1970s in the framework of the so-called theories of galaxy formation which were based on a linear approximation to the field equations. Bonnor had the result earlier and from the exact theory.

Note that from (1) and (4) we can write

$$f(R) = -k(R)R^2, \tag{5}$$

i.e. the curvature index k in the L-T model depends on position. It may thus happen that the L-T universe will be "open" ($k<0$) in one part of the spacetime and "closed" ($k>0$) elsewhere. This shows that the classification of cosmological models into open and closed according to the curvature of their spatial sections applies only to the FLRW class and is forced upon us by the strong *assumptions* about symmetry underlying this class. In more general models such a distinction may be sometimes impossible and so irrelevant in general (note also the last paragraph of section 7.6 below).

From (2) one can conclude that the big bang may have occurred at different times in different locations which is also quite an enlightening departure from the FLRW picture.

Although originally derived in a coordinate-dependent way, the L-T model can be uniquely defined by the following geometrical and physical properties: 1. The spacetime is spherically symmetric, 2. The source in the Einstein's equations is dust + the cosmological term, 3. The expansion and shear are nonzero. This characteristic follows from the work of Collins and Szafron (1979). The model is the limiting case of spherical symmetry and constant pressure ($p = \Lambda$) of the Szafron (1977) solutions. The subcase $\Lambda = 0$ is the spherically symmetric limit of one of the Szekeres (1975) solutions.

The L-T model is the only one among those described in this note that gained a limited recognition and is occcasionally used for discussing cosmological problems.

7.3 The McVittie (1933) model

This solution, called by McVittie "the mass particle in an expanding universe" is a superposition of the Schwarzschild solution and the FLRW models. The metric of the model (in a slightly modified notation) is

$$ds^2 = \left(\frac{1 - M(t,r)}{1 + M(t,r)}\right)^2 dt^2 - \frac{e^{\beta(t)}[1 + M(t,r)]^4}{c^2[1 + \frac{1}{4}kr^2]^2}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{6}$$

where

$$M(t,r) = \frac{\mu(t)}{2r}(1 + \tfrac{1}{4}kr^2)^{1/2}, \tag{7}$$

k and c are constants, $\mu(t)$ is an arbitrary function, and $\beta(t)$ is related to μ by

$$\dot{\beta}\mu = -2\dot{\mu}. \tag{8}$$

In McVittie's original notation k was represented as $1/R^2$ which suggested (incorrectly) that the constant must be positive. The metric (6)–(8) fulfills the Einstein equations with the source being a shearfree expanding perfect fluid that has non-geodesic flowlines (the non-geodesic flow is forced by spatial gradients of pressure).

The Schwarzschild solution in the isotropic coordinates is obtained from (6)–(8) when μ and β are constant and $k = 0$. The FLRW models result when $\mu = 0$ (so $M = 0$), then (8) is fulfilled with arbitrary $\beta(t)$.

This solution looks appropriate for discussing, for example, the following problems:

1. The influence of the expanding universe on the mass of an isolated object. Note that $\mu(t)$ becomes the mass in the Schwarzschild limit, and $e^{\beta/2}$ becomes the scale factor in the FLRW limit. Consequently, (8) implies that the mass of an isolated object decreases with time ($\dot{\mu}<0$) when the universe expands ($\dot{\beta}>0$) which is a very neat Machian effect.
2. The influence of cosmological expansion on the orbits of gravitationally bound systems. The "Swiss cheese" model of Einstein and Strauss (1945) suggested that no such influence should occur. The McVittie model shows that the orbits should be expanding at least due to the decrease of the mass of the central body (see another reason two paragraphs below).
3. The description of black holes in the cosmological background (i.e. asymptotically nonflat black holes). Even the definition of such a black hole is a problem of debate (Ellis 1984) while here we have a rather obvious example for testing the (not yet formulated) theory.

Problem 3 is too modern to have been considered in the 1930s, but 1 and 2 could well have been attempted. McVittie did some preliminary work on problem 2. It was a qualitative discussion, without proper care being taken to define the radius of the orbit in an invariant or measurable way. Noerdlinger and Petrosian (1971) used the McVittie model to discuss, also only qualitatively, the general relativistic correction to the Newtonian effect of the expansion of an orbit due to the outflow of mass from within the orbit (in (6) the cosmic medium extends throughout the planetary system and streams outward; Noerdlinger and Petrosian were actually interested in the orbit of a galaxy in a cluster). No other studies of problem 2 and no studies at all of problem 1 are known to me.

The solution (6)–(8) was guessed rather than derived, and no derivation was presented until today. The model will be shown in section 7.5 to be a subcase of a more general class but the limiting transition has no physical interpretation.

McVittie observed that if the cosmological constant is nonpositive, then "at some time in the past the expansion started instantaneously with a finite velocity" and "there is a 'retarding force' slowing up the expansion which, obviously, cannot be the initial cause that started the latter." McVittie used this as an argument that λ must be positive, but today we would say that it was a prediction of singularity, apparently independent of the earliest singularity theorem by Tolman and Ward in 1932 (Ellis 1988a).

7.4 The Wyman (1946) solution

Wyman (1946) investigated the spherically symmetric solutions of Einstein's equations for which the source was a shearfree expanding perfect fluid obeying a barotropic equation of state. He found the follow-inhomogeneous solution:

$$ds^2 = \tfrac{1}{2}(\alpha C_2 C_3 t + \beta)^{-1}\mathcal{P}^{-2}\mathcal{P}_{,v}^{2}dt^2 - (a\mathcal{P}^2)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\omega^2)], \tag{9}$$

where C_2, C_3, α, β and c care arbitrary constants, $a = 144/c^2$,

$$v \overset{\text{def}}{=} t + \tfrac{1}{2}C_2 r^2, \tag{10}$$

and $\mathcal{P}(v)$ is determined by

$$\mathcal{P}_{,v}^{2} = 4\mathcal{P}^3 - C_3. \tag{11}$$

The physical interpretation of the metric in unknown, but it is a general solution of a formal problem: under the assumptions stated,[1] (9)–(11) is the unique inhomogeneous model. Such a statement is contained in Wyman's paper, but the original proof made it difficult to see how (9)–(11) is singled out from among the other solutions (Krasiński 1988). A transparently complete proof by a different method was published later by Collins and Wainwright (1983).

In the limit of spatial homogeneity, $C_2 \to 0$, only the de Sitter model is obtained from (9)–(11), although the general FLRW models fulfill all the initial assumptions and should be expected to also show up as subcases. Thus the FLRW models are isolated in Wyman's class, with no inhomogeneous parent solutions. This is a consequence of the barotropic equation of state (Krasiński 1988, see also section 7.5).

7.5 The Kustaanheimo–Qvist (1948) spacetimes

Kustaanheimo and Qvist (1948) (abbreviated K–Q) considered the same problem as Wyman, but without the assumption $p = p(\varrho)$. The metric they obtained is

$$ds^2 = [\dot{F}/(A(t)F)]^2 dt^2 - F^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{12}$$

where $A(t)$ is an arbitrary function, $x = r^2$ and $F(t,x)$ obeys

$$F_{,xx}/F^2 = f(x), \tag{13}$$

$f(x)$ being an arbitrary function. For two forms of $f(x)$ general solutions were given later: by Wagh (1955) for $f = \alpha x$ where $\alpha =$ constant and by Wyman (1978) for $f = (ax^2 + bx + c)^{-5/2}$ where a, b and c are constants. Even with these special forms of f, the general solution for F is not an

[1] The assumptions were translated into modern language, the terms *shear* and *expansion* not being in use in 1946.

elementary function. K–Q discussed briefly the problem of solvability of (13) in elementary functions, and a more systematic investigation was done much later by Stephani (1983). The conditions given by K–Q and Stephani are sufficient, but not necessary, so they do not exhaust all the possibilities. The class defined by (12)–(13) includes as subcases, among others:

1. The FLRW models in which $f = 0$ and $F = (1 + \frac{1}{4}kr^2)/R(t)$.
2. The Wyman solution of section 7.4 in which $f = \frac{3}{2}C_2^2 = \text{const}$ and $F(t, x) = |\mathscr{P}|(v)$, $v = t + \frac{1}{2}C_2 x$.
3. The McVittie solution of section 7.3 in which $f = -\frac{3}{4}(\mu e^{\beta/2}/c)(x + \frac{1}{4}kx^2)^{-5/2}$ and $F = ce^{-\beta/2}(1 + \frac{1}{4}kx)[1 + M(t,x)]^{-2}$ (note that by (8), $\mu e^{\beta/2}$ is a constant).
4. The spherically symmetric limit of the Stephani model of section 7.6 in which $f = 0$ and F is the same as in FLRW, but $k = k(t)$ is an arbitrary function.

The limiting transition 2 is equivalent to imposing the barotropic equation of state on inhomogeneous solutions of (13). Transition 4 is equivalent to setting the conformal curvature to zero, and transition 1 results when pressure is homogeneous in addition. Transition 3 does not seem to have any clear interpretation.

The K–Q paper is a rare example of a problem satisfactorily treated already at first go. The authors reduced the Einstein's equations in their case to the simple (13). After forty years their result is still at the top of a large family of models, and no far-reaching generalizations were found (Barnes 1973 obtained the plane- and hyperbolically symmetric counterparts of (12)–(13), and Faulkes (1969) found the generalization of (13) in which a spherically symmetric electromagnetic field is present). Unfortunately, the paper was published in an unknown journal and the result did not reach anyone for a long time. Over the years, the K–Q equation (13) was rederived repeatedly by several authors (see list in section 7.7). Some of the particular solutions of (13) were discussed as stellar models. The only cosmological application of a subcase of (12)–(13) that I know about is the one discussed in section 7.3.

Note that, $f(x)$ given, the general solution of (13) will contain two arbitrary functions of time as integration "constants." Hence, with arbitrary f the family of solutions of (13) is labelled by three arbitrary functions, each of one variable. All this wealth of solutions shrinks to (9)–(11) plus the FLRW models when the barotropic equation of state is imposed. This shows how restrictive the assumption $p = p(\varrho)$ is.

7.6 The Stephani (1967) universe

Stephani (1967) found the following simple generalization of the FLRW models:

$$ds^2 = D^2(t,x,y,z)dt^2 - R^2(t)V^{-2}(t,x,y,z)(dx^2 + dy^2 + dz^2) \tag{14}$$

where

$$V = 1 + \tfrac{1}{4}k(t)[(x - x_o(t))^2 + (y - y_o(t))^2 + (z - z_o(t))^2], \tag{15}$$

$$D = F(t)\left(\frac{R}{V}\right)\left(\frac{V}{R}\right)_{,t} \tag{16}$$

and $R(t)$, $k(t)$, $x_o(t)$, $y_o(t)$, $z_o(t)$ and $F(t)$ are arbitrary functions. The matter-density and pressure are given by

$$\varkappa\varrho = 3C^2(t), \tag{17}$$

$$\varkappa p = 3C^2 + 2C\dot{C}\frac{V}{R}\bigg/\left(\frac{V}{R}\right)_{,t}, \tag{18}$$

where $C(t)$ is a function related to k, F and R by

$$k = (C^2 - 1/F^2)R^2. \tag{19}$$

This is the most general conformally flat solution with an expanding perfect fluid source (Kramer et al. 1980). It was found in this generality only in 1967, but its spherically symmetric subcase (which is the limit $f = 0$ of the K–Q models and results from (14)–(16) when x_o, y_o and z_o are all constant) appeared already in the papers by Wyman (1946) and K–Q (1948), and was rediscovered many times more (see list in section 7.7). The FLRW models result when x_o, y_o, z_o and k are constant; then D becomes a function of t only and may be scaled to 1 by transforming t. After such a transformation $F = -R/\dot{R}$ and (19) coincides with the Friedmann equation.

Since $k = k(t)$, the model predicts that the universe can have its spatial curvature negative at one time t_1 (when $k(t_1) < 0$) and positive at another time t_2 (when $k(t_2) > 0$). This model is thus complementary to the L-T solution where the spatial curvature index varies in space (see the remark in the paragraph after (5)). This intriguing property of the Stephani model was pointed out in several earlier papers of the present writer, but as yet the solution has not been seriously considered as a viable model of our observed universe. The spherically symmetric subcase was repeatedly used as a model of stellar collapse.

7.7 Some historical curiosities

The previous sections presented only those papers which, in the present writer's opinion, were significant though not properly appreciated contributions to theoretical cosmology. The same solutions, or their subcases, were rediscovered later, however, more than once in each case. Such rediscoveries are listed here, but the lists are not guaranteed to be complete. The papers which make reference to previous discoveries are omitted deliberately.

The $\Lambda = 0$ subcase of the L-T model from section 7.2 was reobtained by Datt (1938). Four of the five special cases listed by Datt are trivial: (a) is the Minkowski metric in spherical coordinates, (b) is the Minkowski metric in a more clever disguise, (c) is the flat FLRW model and (d) is a simple coordinate transform of (b). Case (e) is the metric which later appeared in the paper by Oppenheimer and Snyder (1939) (see below). Datt's section 7 introduces one more solution which is a subcase of the Szekeres (1975) solutions and a generalization of the spherical Kantowski–Sachs (1966) solution to inhomogeneous matter-density. The L-T model with $\Lambda = 0$ was rederived once again by Carr and Hawking (1974).

Oppenheimer and Snyder (1939) investigated the gravitational collapse of a spherical cloud of dust. Their physical discussion of the process was a new result, but the metric they derived is the $\Lambda = 0 = f$ limit of the L-T solution and coincides with Datt's (1938) case (e) (they instantly specialized the metric further so that the flat FLRW model surrounded by Schwarzschild space resulted).

The Kustaanheimo–Qvist equation (13) was for the first time obtained (in different variables) by Wyman (1946), and then rederived by Wagh (1955, in Wyman's variables), Taub (1968, in still other variables), Cahill and McVittie (1970, in yet other variables), Barnes (1973), Wyman (1976), Glass (1979) and Banerjee and Chakravarty (1979). It should be noted that Barnes (1973) found *all* rotation-free shearfree perfect fluid solutions to Einstein's equations, among them the plane- and hyperbolically symmetric counterparts of the Kustaanheimo–Qvist spacetimes and the Stephani (1967) model. Wyman (1976) presented a wealth of solutions of the K–Q equation. Banerjee and Chakravarty (1979) reobtained the plane-symmetric models of Barnes (1973) (this may be difficult to reognize) and the Stephani (1967) model, but all under more restrictive assumptions. The papers dealing with various particular solutions of the K–Q equation (13), corresponding to different $f(x)$ are too numerous to be listed here (see Krasiński 1989 for a partial list). Also omitted are particular solutions with electric charge (see Sussman 1987 and 1988).

The spherically symmetric subcase of the Stephani (1967) model of section 7.6 was for the first time casually mentioned (and instantly dismissed because it does not admit $p = p(\varrho)$) by Wyman (1946), and then reobtained by Kustaanheimo and Qvist (1948, their case 1), Raychaudhuri (1955), Gupta (1959), Bondi (1967), Thompson and Whitrow (1967), Taub (1968), Cahill and McVittie (1970), Cook (1975), Glass (1979) and Pandey, Gupta and Sharma (1983). Some more authors discussed still simpler subcases of the model.

7.8 Conclusion

The foregoing sections show that inhomogeneous cosmological models have been since more than fifty years a subject of scientific activity which produced interesting and valuable results. Unfortunately, they were mostly ignored (with the possible exception of the L-T model) because most cosmologists *knew* that our actual universe was homogeneous. For this reason, the papers mentioned in this note did not really count in making history of cosmology. A symposium such as this one is perhaps the right place to recall them and in this way, at least, prevent further rediscoveries.

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Discussion

Ehlers:
Could you please state again the assumptions underlying Wyman's solution?

Krasiński:
It follows uniquely from the following assumptions: 1) The spacetime is spherically symmetric, 2) The source is a perfect fluid, 3) Shear is zero, 4) Expansion is nonzero, 5) The barotropic equation of state holds, 6) The spacetime is inhomogeneous.

Ehlers:
But then it does not seem to reproduce the static spherically symmetric solutions.

Krasiński:
This is correct. In the process of integrating the Einstein equations for a spherically symmetric metric with a perfect fluid source, alternatives occur at which we must choose between mutually exclusive possibilities. One such choice is: either $g_{11,t} \neq 0$ and then $g_{oo} = f(t)[(\ln g_{11})_{,t}]^2$, or else a static solution results. From this point on, if we follow the first possibility, we lose control of the static case. Similarly, at a later point, another alternative occurs: either $(g_{11}^{-1/2})_{,uu} \stackrel{\text{deF}}{=} F = 0$ where $u = r^2$ (then the FLRW models result) or $F \neq 0$ and the Wyman solution follows. These possibilities are mutually exclusive, too: in the limiting case of spatial homogeneity, the

Wyman solution reproduces only the de Sitter spacetime, the general FLRW models are not contained in it, even though they fulfil assumptions 1 to 5.

Ellis:
1) De Sitter wrote down field equations for Lemaître–Tolman in 1917.
2) Guy Omer investigated inhomogeneous cosmologies in 1949.

Krasiński:
Omer investigated the Lemaître–Tolman solution. I did not take into account all papers written on inhomogeneous models, I only tried to present the different inhomogeneous solutions and a selection of papers in their interpretation.