Shear-free normal cosmological models

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Shear-free normal cosmological models are the perfect fluid solutions of Einstein's equations in which rotation and shear vanish, and which are not static [they were all found by A. Barnes, Gen. Relativ. Gravit. **4**, 105 (1973)]. They are either spherically, plane, or hyperbolically symmetric. Their symmetries are discussed in various coordinate systems and related to the conformal group of the three-dimensional flat space. A coordinate representation is introduced which unites all three cases into a single two-parameter family. The limiting transitions to the Friedman–Lemaitre–Robertson–Walker (FLRW) models and to the Schwarzschild–de Sitter-like solutions are presented.

I. WHAT ARE THE SHEAR-FREE NORMAL COSMOLOGICAL MODELS AND WHY ARE THEY INTERESTING?

All the perfect fluid solutions of Einstein's equations in which rotation and shear vanish were found by Barnes.[1] Some of them are static and thus of no interest in cosmology. One of the nonstatic solutions is conformally flat; it was found earlier by Stephani[2] and studied by this author elsewhere.[3,4] The remaining solutions of Barnes, which are of Petrov type D and nonstatic, qualify as inhomogeneous models of the Universe and will be called here "the Barnes cosmological models."

There are three classes of them: spherically, plane, and hyperbolically symmetric [the first class was in fact discovered by Kustaanheimo and Qvist[5] (KQ) in 1948 and reobtained a few times more,[6–9] also the plane symmetric model was later rediscovered[9]]. Because they have three-dimensional symmetry groups acting on two-dimensional spacelike orbits, all structures in them have one spatial dimension. This is too simple to describe the three-dimensional structures observed in the real Universe, but may be the first step in the right direction. As argued by this author,[10–12] in the plane symmetric Barnes model it seems possible to set up initial conditions in such a way that the matter density is a periodic function of the (invariant) spatial distance. It would then be an example of a model with a discrete symmetry group,[13] combining large scale homogeneity with small scale inhomogeneity.[14] This is all the more attractive because, as will be shown further in this paper, all the classical Friedman–Lemaitre–Robertson–Walker (FLRW) cosmological models are contained in the Barnes models in the limiting case of spatially homogeneous matter distribution. Hence the Barnes models represent an inhomogeneous perturbation (within the exact theory) superimposed on the FLRW background, and are capable of reproducing the classical cosmological results in the limit.

Since they are too simple for cosmological purposes, further generalizations are needed. A study of geometrical properties of the Barnes models may thus be useful. In this paper, a convenient representation of the three Barnes's classes (Sec. III) is used in which each class is generated by the same differential equation. Symmetry groups of the three classes (Sec. V) are investigated, and special subcases possessing four-dimensional groups are revealed (Sec. VI, all but one of them are vacuum solutions with Λ). Since the symmetries of the models are closely related to the conformal group of the three-dimensional Euclidean space, an account of properties of this group is given (Sec. IV). A new coordinate representation is introduced (Sec. VII) which unites the three classes into a single two-parameter family in which any one model can be continuously deformed into any other. Symmetries of the three models in these coordinates are presented in Sec. VIII. Finally, it is shown in Sec. IX how the FLRW models result as limiting cases of the Barnes models.

In order to make this paper self-contained, an account of Barnes's original results is given in Sec. II.

The readers should be aware that apart from the Barnes models, other generalizations of the FLRW models are also found in the literature. Most important are the geodesic and shearing perfect fluid models of Szafron[15] which generalize those of Szekeres[16] and Lemaitre,[17–20] and the Petrov type N perfect fluid models of Oleson.[21] Many more papers were published, but most of them deal with special cases of those mentioned here and sometimes they duplicate each other. A detailed survey displaying the interdependences between the various models is being prepared by this author. Readers wishing to contribute to the list are welcome to do so.

II. THE MODELS AS OBTAINED BY BARNES

Of the several solutions found by Barnes we shall consider here only those which have the expansion scalar and the Weyl tensor both nonzero. In the table of Ref. 1 they are contained in the lines IBE and IIE, but those from IBE are special cases of the latter and need not be considered separately. The metric in those solutions [changed to signature $(+---)$] is

$$ds^2 = Y^{-2}(t,r)\left[\left(\frac{Y_{,t}}{3\Theta}\right)^2 dt^2 - \frac{dr^2}{r^2} - d\theta^2 - f^2(\theta)d\phi^2\right], \tag{2.1}$$

where $\Theta(t)$ (the expansion scalar) is an arbitrary function, $Y(t,r)$ is given by the equation

$$r^2 Y_{,rr} + rY_{,r} - KY = b(r)Y^2, \tag{2.2}$$

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$b(r)$ is another arbitrary function, $K = +1, 0,$ or -1, and

$$f(\theta) = \begin{cases} \sin\theta, & \text{for } K = +1, \\ \theta, & \text{for } K = 0, \\ \sinh\theta, & \text{for } K = -1. \end{cases} \qquad (2.3)$$

The model with $K = +1$ is spherically symmetric, the one with $K = 0$ is plane symmetric, and the one with $K = -1$ is hyperbolically symmetric. The constant K should not be confused with the FLRW curvature index k; the former is the sign of curvature of the two-dimensional orbits of the symmetry groups mentioned above.

The source is in each case a perfect fluid, and the coordinates in (2.1) are comoving; thus the velocity field is

$$u^\alpha \underset{*}{=} (3\Theta Y/Y_{,t})\delta^\alpha_0. \qquad (2.4)$$

From the form (2.1) it is not easy to reobtain the FLRW models, so we shall change to another parametrization.

III. COORDINATES BETTER ADAPTED TO THE FLRW LIMIT

Let $R(t)$ be a function (which will coincide with the scale factor in the FLRW limit), and let

$$F(t) = -1/(3\Theta). \qquad (3.1)$$

In the spherically symmetric case we then have

$$ds^2 = D^2\,dt^2 - [R(t)/V(t,x,y,z)]^2(dx^2 + dy^2 + dz^2), \qquad (3.2)$$

where

$$D = F(R/V)(V/R_{,t}), \qquad (3.3)$$

$$u \overset{\text{def}}{=} r^2, \qquad (3.4)$$

$$(x,y,z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad (3.5)$$

$$Y(t,r) = V(t,u)/[rR(t)], \qquad (3.6)$$

and the function $V(t,u)$ is determined by the equation

$$R(t)w_{,uu}/w^2 = f(u), \qquad (3.7)$$

where $f(u)$ is an arbitrary function, and, in the present case,

$$w = V(t,u), \qquad (3.8)$$

$$b(u) = f(u)u^{5/2}. \qquad (3.9)$$

Equation (3.7) is the Kustaanheimo–Qvist equation.[5]

The FLRW models (all of them) are obtained from here when $f = 0$ and $V_{,t} = 0$, thus $V = 1 + (1/4)kr^2$, where $k = \text{const}$. Without the assumption $V_{,t} = 0$ the spherically symmetric subcase of the Stephani universe[2–4] results where $k(t)$ is an arbitrary function.

The matter density and pressure are here equal to

$$\kappa\rho = \frac{3}{F^2} + \frac{8ufV^3}{R^3} + \frac{12VV_{,u}}{R^2} - \frac{12uV^2_{,u}}{R^2}, \qquad (3.10)$$

$$\kappa p = -\frac{3}{F^2} + \frac{4(uV^2_{,u} - VV_{,u})}{R^2} - \frac{2F_{,t}}{(F^2D)} + 4\left[\frac{F}{(R^2D)}\right]\left(1 - \frac{2uV_{,u}}{V}\right)(VV_{,tu} - V_{,t}V_{,u}) \qquad (3.11)$$

($\kappa = 8\pi G/c^4$). In the FLRW limit, Eqs. (3.10) and (3.11) reproduce the Friedman equations if the t coordinate is chosen so that $F = -R/R_{,t}$.

In the plane symmetric case the metric form is again (3.2) and (3.3) where this time

$$u = z, \qquad (3.12)$$

$$(x,y,z) = (\theta\cos\phi, \theta\sin\phi, \ln r), \qquad (3.13)$$

$$Y(t,r) = V(t,z)/R(t), \qquad (3.14)$$

and again V is determined by (3.7) and (3.8) where

$$b(u) = f(u). \qquad (3.15)$$

When $f = 0$ and $V = 1$, the flat FLRW model results. However, with $f = 0$ and $V = a + bz$; $a,b = \text{const}$, the open ($k < 0$) FLRW model is obtained in a nontrivial disguise, as will be shown in Sec. IX. The matter density and pressure are in this case[22–25]

$$\kappa\rho = \frac{3}{F^2} + \frac{2fV^3}{R^3} - \frac{3V^2_{,z}}{R^2}, \qquad (3.16)$$

$$\kappa p = -\frac{3}{F^2} + \frac{V^2_{,z}}{R^2} - \frac{2F_{,t}}{(F^2D)} - 2\left[\frac{FV_{,z}}{(R^2DV)}\right](VV_{,tz} - V_{,t}V_{,z}). \qquad (3.17)$$

Finally, in the hyperbolically symmetric case, the metric form can be transformed into (3.2) and (3.3) where

$$u = x/y, \qquad (3.18)$$

$$V(t,x,y) = yw(t,u), \qquad (3.19)$$

and $w(t,u)$ obeys (3.7). The transformations corresponding to (3.5) and (3.6) and also (3.13) and (3.14) are carried out in two steps. We first observe[26,27] that the two-dimensional metric $(d\theta^2 + \sinh^2\theta\,d\phi^2)$ may be transformed into $(d\tau^2 + e^{2\tau}\,dz^2)$ (see Appendix A), so (2.1) can be written as

$$ds^2 = F^2(t)(Y_{,t}/Y)^2\,dt^2 - r^{-2}Y^{-2}(t,r)(dr^2 + r^2\,d\tau^2 + r^2e^{2\tau}\,dz^2). \qquad (3.20)$$

Now we define

$$r = \exp\{\arcsin[(x/y)^2 + 1]^{-1/2}\}, \qquad (3.21)$$

$$\tau = -\tfrac{1}{2}\ln(x^2 + y^2), \qquad (3.22)$$

$$Y = [(x/y)^2 + 1]^{-1/2}w(t,u)/R(t), \qquad (3.23)$$

and change (3.20) into (3.2) with (3.3), (3.19), and (3.18). Moreover, if variables are changed in (2.2) according to (3.21), (3.18), and (3.19), then w will obey (3.7) where this time

$$b(u) = f(u)(u^2 + 1)^{5/2}. \qquad (3.24)$$

With $f = 0$ and $w_{,t} = 0$, this Barnes model can reproduce the flat and the open FLRW model, but the limit is not achieved trivially; see Sec. IX. The matter density and pressure are here[22]

$$\kappa\rho = \frac{3}{F^2} + \frac{2(u^2+1)fw^3}{R^3} + \frac{6uww_{,u}}{R^2} - \frac{3(u^2+1)w^2_{,u}}{R^2} - \frac{3w^2}{R^2}, \qquad (3.25)$$

$$\kappa p = -\frac{3}{F^2} + \frac{(u^2+1)w_{,u}^2}{R^2} - \frac{2uww_{,u}}{R^2} - \frac{2F_{,t}}{(F^2D)} + \frac{w^2}{R^2} + 2\left[\frac{F}{(R^2D)}\right] \times \left[u - \frac{(u^2+1)w_{,u}}{w}\right](ww_{,tu} - w_{,t}w_{,u}). \quad (3.26)$$

The transformations changing (2.2) into (3.7) fulfill in each case the Barnes equations (6.12), (6.13), and those following them.

In the previous papers of this author,[10-12] the Barnes models were unknowingly reobtained and some were named differently. The hyperbolically symmetric model was called there "line homogeneous." The "axially symmetric universe" of Ref. 11 is the unified representation from Sec. VII of this paper, and the other cases of Ref. 11 lead to one or another of the three Barnes models.

Several special solutions of Eq. (3.7) were found by different authors[5,6,8,28-37] (the list of references is not guaranteed to be complete), and the existence of solutions consisting of elementary functions was systematically investigated by Stephani.[38] The result was that Eq. (3.7) can be integrated to a first-order equation when $f(u) = u^n$ or $f(u) = e^u$ or $f(u) = (u+\alpha)^n(u+\beta)^{-n-5}$ (α,β,n are constants), and it can be completely solved in elementary functions when $f(u) = u^{-15/7}$ or $f(u) = (au^2 + 2bu + c)^{-5/2}$ (a,b,c are constants). For this last case Wyman[30] provided a general formal solution (not necessarily elementary). In the coordinates used in this section, all these results translate immediately into the corresponding plane and hyperbolically symmetric cases even though the papers quoted were concerned with spherically symmetric space-times. However, as argued in Refs. 10 and 11, it may be more important from the physical point of view to solve Eq. (3.7) in the variable $l(u,t)|_{t=t_0}$, where l is the affine parameter on the geodesics orthogonal to the fluid flow and to the group orbits, since l has a direct geometrical meaning while u has not. The transformation $u \to l(u)$ may not be elementary.

The transformations that change (2.1) into (3.2) preserve in each case the comoving character of the coordinate system, and so

$$u^\alpha \underset{*}{=} D^{-1}\delta_0^\alpha. \quad (3.27)$$

The Weyl tensor[22] is in each case proportional to the arbitrary function $f(u)$, and so will vanish whenever $f(u) = 0$. Then, each of the Barnes models becomes a subcase of the Stephani universe.[2-4] With $f \neq 0$, the Barnes models are of Petrov type D. Note that the Stephani universe has in general no symmetry,[3] so only its special cases are contained as limits $f \to 0$ in the Barnes models.

Since the space metric in (3.2) is manifestly conformally flat, its symmetries will be closely related to conformal symmetries of the flat space. These are shortly described in the next section.

IV. THE CONFORMAL GROUP OF A FLAT SPACE

In this section we shall consider a flat Riemannian space of arbitrary signature and arbitrary dimension n. Let x^A, $A = 1,\dots,n$, be the Cartesian coordinates so that the metric form is

$$ds_x^2 = \epsilon_1(dx^1)^2 + \cdots + \epsilon_n(dx^n)^2, \quad (4.1)$$

where each ϵ_i equals either $+1$ or -1. A transformation of coordinates $x \to y^A(x)$ is called a conformal symmetry of (4.1) if it changes the metric form (4.1) to

$$ds_y^2 = \Phi(y)ds_{x\to y}^2, \quad (4.2)$$

where Φ is a function and $ds_{x\to y}^2$ is obtained from (4.1) by replacing all x^A by y^A. The conformal group (i.e., the group of conformal symmetries) of an n-dimensional flat space has $[\frac{1}{2}n(n+1)+1]$ parameters; $\frac{1}{2}n(n-1)$ of them belong to the symmetry group (for which $\Phi = 1$) and 1 belongs to the dilatation transformation, $x^A/y^A = l = \text{const}$ (where $\Phi = l^{-2}$). The remaining n parameters are connected with the following transformations:

$$x^A = (y^A + C^A y_R y^R)/T,$$
$$T \overset{\text{def}}{=} 1 + 2C_S y^S + C_S C^S y_P y^P, \quad (4.3)$$

where C^A, $A = 1,\dots,n$, are the group parameters. After Plebański[39] we shall call (4.3) the Haantjes transformations, although Haantjes derived in fact only the special cases of (4.3) where (1) $C_S C^S = 0$ (Ref. 40) and (2) just one C^A was nonzero.[41] The group (4.3) is Abelian. Composing two such transformations with the sets of parameters C^A and D^A results in a single transformation (4.3) with the set of parameters $(C^A + D^A)$. Consequently, the inverse transformation to (4.3) is obtained by interchanging x's with y's and replacing all C^A by $(-C^A)$. The generators of (4.3) are

$$J_B = y_R y^R \frac{\partial}{\partial y^B} - 2y_B y^R \frac{\partial}{\partial y^R}. \quad (4.4)$$

The following properties of (4.3) are useful in calculations:

$$x_A x^A = y_A y^A/T, \quad (4.5)$$
$$dx_A\, dx^A = dy_A\, dy^A/T^2. \quad (4.6)$$

Equation (4.3) can be interpreted as the succession of the following three transformations.[39]

(1) Inversion in the (pseudo-) sphere of radius L centered at $x^A = 0$, $x^A = L^2u^A/u_S u^S$.

(2) Translation by the vector L^2C^A, $u^A = w^A + L^2C^A$.

(3) Inversion in an identical pseudosphere centered at $w^A = 0$, $w^A = L^2y^A/y^S y_S$.

Since L cancels in the end, it may be assumed that $L = 1$ without loss of generality. In the following we shall often denote (4.3) by

$$x^A = H(C^1,\dots,C^n)y^A; \quad (4.7)$$

its inverse is then $y^A = H(-C^1,\dots,-C^n)x^A$.

V. SYMMETRY GROUPS OF THE BARNES MODELS

For the spherically symmetric and the plane symmetric model, the symmetries are well known. The generators are $J_i = \epsilon_{ijk} x^j (\partial/\partial x^k)$ in the former case and $J_1 = \partial/\partial x$, $J_2 = \partial/\partial y$, $J_3 = x(\partial/\partial y) - y(\partial/\partial x)$ in the latter. For the hyperbolically symmetric model in the form given by (3.2), (3.3), (3.7), (3.18), and (3.19) the generators are found from the Killing equations to be

$$J_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{5.1}$$

$$J_2 = \frac{\partial}{\partial z}, \tag{5.2}$$

$$J_3 = -2xz\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial y} + (x^2+y^2-z^2)\frac{\partial}{\partial z}. \tag{5.3}$$

The corresponding symmetry transformations are, respectively,

$$x'^i = lx^i, \quad i = 1,2,3, \quad l = \text{const}, \tag{5.4}$$

$$z' = z + a, \quad a = \text{const}, \quad (x',y') = (x,y), \tag{5.5}$$

and the Haantjes transformation,

$$(x',y',z') = H(0,0,C)(x,y,z), \quad C = \text{const}, \tag{5.6}$$

which will be for once written out explicitly:

$$x' = \frac{x}{T_H}, \quad y' = \frac{y}{T_H}, \quad z' = \frac{[z + C(x^2+y^2+z^2)]}{T_H},$$
$$T_H = 1 + 2Cz + C^2(x^2+y^2+z^2). \tag{5.7}$$

It may be verified that the algebras of the symmetry groups of the three models are of Bianchi types IX, VII_0, and VIII, respectively. These are all the Bianchi types possible with two-dimensional orbits.

As usual, while solving the Killing equations several alternatives are encountered, of the form: either a certain differential expression vanishes and a constant in the Killing vector survives, or else the constant vanishes and a possible one-parameter symmetry group is absent. In this way, special subcases of a class of metrics are revealed that have higher symmetry. Such special cases of the Barnes models will be presented in the next section. However, a few more cases show up in several places which will be ignored for the reasons explained below.

(1) If $V_{,t} = 0$, then Eq. (3.7) implies that either (a) $R_{,t} = 0$, and Eq. (3.3) becomes invalid, or (b) $w_{,uu} = 0$. Equation (3.3) results from the Einstein equations[22] $G_{0i} = 0$, $i = 1,2,3$, for the metric (3.2) if $(V/R)_{,t} \neq 0$. Otherwise, $G_{0i} \equiv 0$ and D remains arbitrary. However, then matter density does not depend on time and $\Theta = 0$ [see Eq. (7.2) in Ref. 14]. Thus case 1(a) is of no interest for cosmology and will be ignored here. In case 1(b) a FLRW model results (see Sec. IX) whose symmetries are well known.

(2) If V separates, $V = g(t)v(x,y,z)$, then Eq. (3.7) implies again that either $(g/R)_{,t} = 0$, which is equivalent to case 1(a), or else case 1(b) occurs.

(3) If $f(u) = 0$, then a subcase of the Stephani universe results (see Sec. IX) that is beyond the scope of this paper.

These subcases will be called "trivial."

VI. SUBCASES OF HIGHER SYMMETRY

A. The spherically symmetric model

A detailed analysis of the Killing equations shows that only one subcase of higher symmetry exists here which is in fact also trivial, but different from those mentioned above. The subcase results when the arbitrary function $f(u)$ from (3.7) is $f = B/u^{5/2}$, where $B = \text{const}$ and $w(t,u) = V(t,u)$ is given by

$$\left(\frac{V}{R}\right)_{,u}^2 - \frac{1}{u}\frac{V}{R}\left(\frac{V}{R}\right)_{,u}$$
$$- \frac{2}{3}\frac{B}{u^{5/2}}\left(\frac{V}{R}\right)^3 - \frac{1}{4u}\left(\frac{1}{F^2} - \frac{\Lambda}{3}\right) = 0, \tag{6.1}$$

where Λ is (the cosmological) constant. Equation (6.1) is a first integral of (3.7) and is equivalent to Eq. (6.7) from Ref. 42 (the correspondence is $V/R = \Phi$, $u = r^2$, $B = f$, $1/F = g$). Therefore (6.1) defines the Schwarzschild–de Sitter solution which in the more familiar "standard" spherical coordinates has the form

$$ds^2 = P\,dt^2 - P^{-1}\,dr^2 - r^2\,d\theta^2$$
$$- (r^2/K)\sin^2(K^{1/2}\theta)d\phi^2, \tag{6.2}$$

where $P \overset{\text{def}}{=} K - 2m/r - \frac{1}{3}\Lambda r^2$, $m = -\frac{4}{3}B$, and $K = +1$ (for the proof see Ref. 42). The reason for the peculiar notation will become clear further on. The additional symmetry is time independence in the coordinates of (6.2).

B. The plane symmetric model

Two subcases of higher symmetry exist here, one of which is again trivial in the same sense as (6.1). In the trivial case the function $f(u)$ from (3.7) is $f = C = \text{const}$ and $w = V$ is given by the following first integral of (3.7):

$$\left(\frac{V}{R}\right)_{,z}^2 - \frac{2}{3}C\left(\frac{V}{R}\right)^3 - \frac{1}{F^2} + \frac{1}{3}\Lambda = 0. \tag{6.3}$$

As indicated in Appendix B, this case can be transformed into (6.2) with $K = 0$, i.e., is the plane symmetric analog of the Schwarzschild–de Sitter solution. The additional symmetry is again time independence in the form (6.2).

The other subcase of higher symmetry has $F = \text{const}$ and

$$V = R(t)azv(azt), \tag{6.4}$$

where a is an arbitrary constant and $v(X)$ is defined by

$$X^2 v_{,XX} + 2Xv_{,X} = Bv^2, \tag{6.5}$$

where $X = azt$, B is another arbitrary constant, and the function $f(u)$ from (3.7) is

$$f = B/(az^3). \tag{6.6}$$

The metric is

$$ds^2 = (azv_{,X}/v)^2\,dt^2 - (azv)^{-2}(dx^2 + dy^2 + dz^2). \tag{6.7}$$

With (6.6), Eqs. (3.7) and (6.5) are consistent. The additional symmetry is

$$t = t'/l, \quad (x,y,z) = l(x',y',z'), \tag{6.8}$$

and its generator

$$J_4 = -t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad (6.9)$$

has the following commutators with the other three generators, $J_1 = \partial/\partial x$, $J_2 = \partial/\partial y$, and $J_3 = x(\partial/\partial y) - y(\partial/\partial x)$:

$$[J_i, J_4] = J_i, \quad i = 1,2; \quad [J_3, J_4] = 0. \quad (6.10)$$

Note that the group generated by J_1, J_2, J_4 is of Binachi type V and has three-dimensional orbits. The orbits are not, however, orthogonal to the flow lines of the fluid and have indefinite geometry (J_4 may be spacelike in one place and timelike in another). Thus (6.7) is a tilted Bianchi type V space-time, with an additional symmetry generated by J_3. Up to inessential reparametrizations, the solution coincides with the one investigated in detail by Collins and Wainwright[43,44] and so has a barotropic equation of state.

C. The hyperbolically symmetric model

As in the spherically symmetric case, the analysis of the Killing equations shows that only one subcase of higher symmetry exists here, in which $w(t,u)$ is given by the following first integral of (3.7):

$$\left(\frac{w}{R}\right)_{,u} = \frac{u}{1+u^2}\frac{w}{R} \pm \left\{\frac{2}{3}\frac{B}{(1+u^2)^{5/2}}\left(\frac{w}{R}\right)^3 - \left[\frac{w}{R(1+u^2)}\right]^2 + \frac{1}{1+u^2}\left(\frac{1}{F^2} - \frac{\Lambda}{3}\right)\right\}^{1/2}, \quad (6.11)$$

where B is a constant given by $f(u) = B/(1+u^2)^{5/2}$, f is the function from (3.7) while Λ is the cosmological constant. By coordinate transformations the solution defined by Eq. (6.11) can be reduced to (6.2) with $K = -1$ [then $\sin(K^{1/2}\theta) = \sin(i\theta) = i\sinh\theta$] and so it is the hyperbolically symmetric counterpart of the Schwarzschild–de Sitter solution. The additional symmetry is time independence in the coordinates of (6.2). The transformation from (6.11) to (6.2) is sketched in Appendix C.

The metrics represented by (6.2) belong to the type D metrics investigated by Kinnersley,[45] and are a subcase of those given by Eq. (25.74) in Ref. 26, but in a different coordinate system.

VII. COORDINATES COVERING ALL THREE MODELS SIMULTANEOUSLY

The following metric represents all the three models of Barnes simultaneously,

$$ds^2 = F^2(t)\left(\frac{R}{S}\right)^2\left(\frac{S}{R}\right)^2_{,t} dt^2 - \frac{R^2(t)}{(z+b)^2 S^2(t,Z)}(dx^2 + dy^2 + dz^2), \quad (7.1)$$

where $F(t)$ and $R(t)$ are arbitrary functions, b is an arbitrary constant, the variable Z is defined by

$$Z = \frac{a - x^2 - y^2 - z^2}{2(z+b)}, \quad (7.2)$$

a is another arbitrary constant, and the function $S(t,Z)$ is defined by the KQ equation,

$$R(t)S_{,ZZ}/S^2 = f_G(Z), \quad (7.3)$$

$f_G(Z)$ being an arbitrary function. With any given $S(t,Z)$, Eqs. (7.1) and (7.2) represent a two-parameter family of metrics. However, we shall show that (7.1)–(7.3) are always equivalent to one of the Barnes models, namely, (I) with $a < b^2$ to the spherically symmetric one, (II) with $a = b^2$ to the plane symmetric one, (III) with $a > b^2$ to the hyperbolically symmetric one.

Thus in the coordinates of (7.1)–(7.3), continuous deformations of one model into another are possible.

A. The spherically symmetric model

We introduce the constant α by

$$a = b^2 - \alpha^2. \quad (7.4)$$

The cases when $a \geqslant 0$ and $a \leqslant 0$ will have to be considered separately. When $a \geqslant 0$, we define the constants A and l by

$$a = A^2(1+2Al)^{-2},$$
$$\alpha = [2l(1+Al)]^{-1}, \quad (7.5)$$

which implies

$$b = (1 + 2Al + 2A^2l^2)/[2l(1+Al)(1+2Al)]. \quad (7.6)$$

The equations are solvable for A and l without further conditions on a and α. Then we perform on (7.1)–(7.3) the sequence of three transformations

(1) $z = z' + A/(1+2Al)$, (7.7)

(2) the Haantjes transformation,

$(x,y,z') = H(0,0,-l)(x',y',z'')$, (7.8)

(3) $z'' = z''' - A$. (7.9)

The final result is transforming Z given by (7.2) into

$$Z = \frac{l(1+Al)}{1+2Al}\frac{u - A^2}{l^2u - (1+Al)^2}, \quad (7.10)$$

where $u = x'^2 + y'^2 + z'''^2$, and the metric (7.1) into

$$ds^2 = D^2 dt^2 - (R/V)^2(dx'^2 + dy'^2 + dz'''^2), \quad (7.11)$$

where $D = F(t)(R/V)(V/R)_{,t}$ and

$$V = [l^2u - (1+Al)^2]S/[2l(1+Al)]. \quad (7.12)$$

This suggests that the substitutions (7.10) and (7.12) change (7.3) into (3.7) and (3.8). This is indeed the case, and $f(u)$ from (3.7) is here

$$f(u) = 2[l(1+Al)]^3[l^2u - (1+Al)^2]^{-5}f_G(Z(u)). \quad (7.13)$$

Note that all the operations make sense also for $a = 0$ (then $A = 0$).

When $a \leqslant 0$, we define A and l by

$$a = -A^2(1 + 2Al + 2A^2l^2)^{-2},$$
$$\alpha = [2l(1+Al)]^{-1}, \quad (7.14)$$

which implies

$$b = (1+2Al)/[2l(1+Al)(1+2Al+2A^2l^2)].$$

Again, (7.14) are solvable for A and l without further conditions on a and α. In the sequence of transformations step (1) is to be replaced by

$$(1') \quad z = z' + A(1+2Al)/(1+2Al+2A^2l^2), \tag{7.15}$$

steps (2) and (3) remaining unchanged. The final result for Z is

$$Z = \frac{l(1+Al)}{1+2Al+2A^2l^2}\,\frac{u+A^2}{l^2u-(1+Al)^2}, \tag{7.16}$$

and for the metric it is again (7.12) and (7.13). Just as before, the substitutions (7.16) and (7.12) change (7.3) into (3.7) and (3.8) with $f(u)$ given again by (7.13). The limit $a \to 0$ (thus $A \to 0$) of (7.14) and the subsequent operations is the same as of (7.5)–(7.13). In either case, each step of the transformation was invertible, so we proved that (7.1)–(7.3) with $a < b^2$ is *equivalent* to the spherically symmetric model for all possible values of a and b.

B. The plane symmetric model

The cases $a = b^2 \neq 0$ and $a = b = 0$ have to be treated separately. In the first case, we perform the sequence of two transformations

$$(4) \quad z = z' + b, \tag{7.17}$$

(5) the Haantjes transformation,

$$(x,y,z') = H(0,0,-1/(2b))(x',y',z''). \tag{7.18}$$

The result on Z in (7.2) is

$$Z = bz''/(z''-2b), \tag{7.19}$$

and the metric (7.1) is transformed into (7.11) with

$$V = (z''-2b)S. \tag{7.20}$$

Just as should be expected, the substitutions (7.19) and (7.20) change (7.3) into (3.7) and (3.8) where this time $u = z''$ and

$$f(z'') = 4b^4 f_G(Z(z''))/(z''-2b)^5. \tag{7.21}$$

When $a = b = 0$, the sequence (4) and (5) should be changed to

$$(4') \quad z = z' + 1, \tag{7.22}$$

$$(5') \quad (x,y,z') = H(0,0,-1)(x',y',z''), \tag{7.23}$$

resulting in

$$Z = \tfrac{1}{2}(z''-1)^{-1} \tag{7.24}$$

and

$$V(t,z'') = (z''-1)S(t,Z(z'')). \tag{7.25}$$

The function $f(z'')$ from (3.7) is here

$$f(z'') = \tfrac{1}{4} f_G(Z(z''))/(z''-1)^5. \tag{7.26}$$

Each step of the transformations is invertible, so (7.1)–(7.3) with $a = b^2$ is *equivalent* to the plane symmetric model for each possible value of a.

C. The hyperbolically symmetric model

We introduce α by

$$a = b^2 + \alpha^2. \tag{7.27}$$

The cases $b > 0$, $b < 0$, and $b = 0$ have to be considered separately. When $b > 0$ we define l_1 and l_2 by

$$b^{-1} = 2l_1^2, \tag{7.28}$$

$$\alpha^{-1} = 2l_1l_2, \tag{7.29}$$

and then perform the following sequence of three transformations:

$$(6) \quad x = (l_1^2 + l_2^2)^{1/2} z'/(l_1l_2), \quad y = x'/l_2 + y'/l_1,$$
$$z = -x'/l_1 + y'/l_2, \tag{7.30}$$

$$(7) \quad x' = x'' + 1/(2l_1), \tag{7.31}$$

$$(8) \quad (x'',y',z') = H(-l_1,-l_2,0)(x''',y'',z''). \tag{7.32}$$

The final result for Z given by (7.2) is

$$Z = au/(bu-\alpha), \tag{7.33}$$

where

$$u = x'''/y'', \tag{7.34}$$

and for the metric (7.1) it is (7.11), where

$$V = y''(bu-\alpha)S(t,Z(u))/a^{1/2}. \tag{7.35}$$

This suggests that the substitution (7.33) together with

$$S = a^{1/2} w(t,Z(u))/(bu-\alpha) \tag{7.36}$$

will change (7.3) into (3.7) with (3.18). This is indeed the case. Thus the metric (7.1)–(7.3) with $b > 0$ and $a > b^2$ can be transformed into the hyperbolically symmetric Barnes model; the function $f(u)$ is then

$$f(u) = \alpha^2 a^{5/2} f_G(Z(u))/(bu-\alpha)^5. \tag{7.37}$$

When $b < 0$, we change (7.28) to

$$b^{-1} = -2l_1^2, \tag{7.38}$$

and the transformation of z in (7.30) to

$$(6') \quad z = x'/l_1 - y'/l_2, \tag{7.39}$$

the other steps in the sequence of transformations remaining unchanged. Instead of (7.33) we then obtain

$$Z = au/(bu+\alpha), \tag{7.40}$$

which corresponds to replacing b by $-b$ and Z by $-Z$ in (7.33). Hence (7.33) also covers, in fact, the case $b < 0$.

The sequence of transformations is different for $b = 0$. Then we define

$$a = \alpha^2 \overset{\text{def}}{=} 1/(4l^2), \tag{7.41}$$

and perform the transformations

$$(9) \quad x = x' + 1/(2l), \tag{7.42}$$

$$(10) \quad (x',y,z) = H(-l,0,0)(x'',y',z'), \tag{7.43}$$

which result in

$$Z = -\alpha u \tag{7.44}$$

with

$$u = x''/y'. \tag{7.45}$$

Equation (7.44) is in fact the limit of (7.33) when $b = 0$. However, the transformations (6)–(8) do not have a meaningful limit $b \to 0$, so the effect of (9) and (10) on the metric (7.1) has to be calculated separately. It turns out that (7.1) changes to (7.11) where

$$V = y' S(t, Z(u)), \tag{7.46}$$

which shows that the resulting metric is the hyperbolically symmetric Barnes model. The function f from (3.7) is here

$$f(u) = \alpha^2 f_G(Z(u)) \tag{7.47}$$

[this is again the limit $b \to 0$ of (7.37)]. Thus (7.1)–(7.3) can be transformed into the hyperbolically symmetric model when $a > b^2$ for each possible value of b. Since each transformation in the sequences (6)–(8) and (9) and (10) is invertible, (7.1)–(7.3) with $a > b^2$ is in fact *equivalent* to the hyperbolically symmetric model.

VIII. SYMMETRIES OF THE BARNES MODELS IN THE COORDINATES OF SEC. VII

Since in the form (7.1)–(7.3) the symmetries are rather difficult to recognize, we shall present them explicitly. Apart from special cases discussed in Sec. VI, the generators of symmetries are

$$J_1 = -2xy \frac{\partial}{\partial x} + (x^2 - y^2 + z^2 + 2bz + a) \frac{\partial}{\partial y} - 2y(z + b) \frac{\partial}{\partial z}, \tag{8.1}$$

$$J_2 = (-x^2 + y^2 + z^2 + 2bz + a) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - 2x(z + b) \frac{\partial}{\partial z}, \tag{8.2}$$

$$J_3 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}. \tag{8.3}$$

The commutation relations are

$$[J_1, J_2] = 4(b^2 - a) J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2. \tag{8.4}$$

Just as the results of Sec. VII suggest, when $a < b^2$, $a = b^2$, or $a > b^2$, the Bianchi type of the algebra (8.4) is IX, VII_0, or VIII, and it corresponds to the symmetry, spherical, plane, or hyperbolic, respectively.

The transformations generated by J_3 are evidently rotations. Those generated by J_1 are generalizations of the Haantjes transformations,

$$\begin{aligned} &x' = x/W, \quad z' = (z + b)/W - b, \\ &y' = [y \cosh(2\beta\tau) + (U/2\beta)\sinh(2\beta\tau)]/W, \end{aligned} \tag{8.5}$$

where τ is the group parameter, $\beta \overset{\text{def}}{=} (a - b^2)^{1/2}$,

$$U \overset{\text{def}}{=} x^2 + y^2 + z^2 + 2bz + a,$$

$$W \overset{\text{def}}{=} [a - x^2 - y^2 - z^2 - 2bz - 2b^2 + U\cosh(2\beta\tau)]/(2\beta^2) + (y/\beta)\sinh(2\beta\tau). \tag{8.6}$$

Equations (8.5) and (8.6) cover all three cases given after Eq. (8.4): when $a < b^2$, β becomes imaginary, so $\beta = iB$ and $\cosh(2\beta\tau) = \cos(2\beta\tau)$, $\beta^{-1}\sinh(2\beta\tau) = B^{-1}\sin(2B\tau)$. With $a = b^2$ we have

$$\lim_{\beta \to 0} W = 1 + 2\tau y + \tau^2(x^2 + y^2 + z^2 + 2bz + b^2),$$

in that case (8.5) and (8.6) is a composition of $z'' = z + b$ and the Haantjes transformation $(x', y', z') = H(0,0,\tau)(x, y, z'')$.

The transformation generated by J_2 is obtained from (8.5) and (8.6) by just interchanging x with y and x' with y'.

IX. THE FLRW LIMIT OF THE BARNES MODELS

Let us change the t coordinate in (3.2) and (3.3) to $T(t)$ defined by

$$\frac{dT}{dt} = -\frac{FR_{,t}}{R}. \tag{9.1}$$

Then, in the new coordinates

$$D = -R^2 (V/R)_{,T}/(VR_{,T}) = -V_{,T} R/(VR_{,T}) + 1. \tag{9.2}$$

Thus the T coordinate was chosen so as to make $F(T) = -R(T)/R_{,T}$. In what follows, we shall use just this coordinate.

In order to obtain the FLRW models from the Barnes models, we must first make the latter conformally flat. As was stated after (3.27), this happens when the function $f(u)$ from (3.7) vanishes. Solving (3.7) in that case we obtain

$$w(t,u) = a(t) + b(t)u. \tag{9.3}$$

With arbitrary a and b the metric corresponding to (9.3) is in each of the three cases a subcase of the Stephani universe[2–4] [in order to verify this, one has to occasionally use transformations like (9.7), (9.9), or (9.10) below].

Now the three models have to be considered separately. In the spherically symmetric case, Eq. (9.3) implies that V in (3.2) will be

$$V = a(T) + b(T)(x^2 + y^2 + z^2). \tag{9.4}$$

If $a \neq 0$, then it can be scaled to 1 by redefining $b(T)$ and $R(T)$. Let us then assume $a = 0$ first. Then obviously $b \neq 0$, and so b can be scaled to 1 by redefining $R(T)$. From (3.10) and (3.11) we then see that $\rho = \rho(t)$ and $p = p(t)$, i.e., (9.4) with $a = 0$, $b = 1$ should be a FLRW model. Indeed, the standard form of the FLRW metric,

$$ds^2 = dT^2 - [R(T)/V]^2(dx^2 + dy^2 + dz^2) \tag{9.5}$$

with

$$V = 1 + \tfrac{1}{4}k(x^2 + y^2 + z^2) \tag{9.6}$$

is obtained in this case by the sequence of two transformations,

$$\begin{aligned} &(1) \quad z = z' - 1, \\ &(2) \quad (x, y, z') = H(0,0,1)(x', y', z''). \end{aligned} \tag{9.7}$$

The resulting $k = 0$.

With $a \neq 0$, we scale a to $+1$ and find from (3.11) that $p_{,u} = 0$ implies either $b = \text{const}$ or $\rho = -p = \text{const}$. The latter case is the de Sitter solution, the former covers all the FLRW metrics ($k = 0, \pm 1$). Thus (9.4) leads to (any) FLRW model when a and b are both constants.

In the plane symmetric case (9.3) implies for (3.2)

$$V = a(T) + b(T)z. \tag{9.8}$$

If $b=0$, then this is evidently the flat FLRW model. Any $b\neq 0$ can be scaled to 1 by redefining $R(T)$. In that case we find from (3.17) that $p_{,z}=0$ implies either $a=\text{const}$ or $\rho=-p=\text{const}$. The latter case is again the de Sitter solution, while the former is a FLRW model. The transformations to (9.5) and (9.6) are the following. When $a\neq 0$,

$$\begin{aligned}&(3)\ (x,y,z)=H(0,0,-1/(2a))(x',y',z'),\\&(4)\ (x',y',z')=a(x'',y'',z'');\end{aligned}\tag{9.9}$$

and, when $a=0$,

$$\begin{aligned}&(5)\ (x,y,z)=H(0,0,\tfrac{1}{4})(x',y',z'),\\&(6)\ z'=z''-2.\end{aligned}\tag{9.10}$$

In both cases the resulting k is necessarily -1. Hence the plane symmetric Barnes model can reproduce only the flat $(k=0)$ and the open $(k=-1)$ FLRW models.

Finally, in the hyperbolically symmetric case, (9.3) implies for (3.2),

$$V=a(T)y+b(T)x.\tag{9.11}$$

If either $a=0$ or $b=0$, then (9.11) is equivalent to (9.8) with $a=0$. We shall thus consider the case $ab\neq 0$. Then a can be scaled to 1 redefining $R(T)$, and from (3.26) we conclude that $p_{,u}=0$ implies either $b=\text{const}$ or $\rho=-p=\text{const}$. The latter is once more the de Sitter solution, while the former is a FLRW metric. It can be transformed to the standard form (9.5) and (9.6) by the following sequence of transformations:

$$\begin{aligned}&(7)\ (x,y,z)=H(-\tfrac{1}{2},b,0)(x',y',z'),\\&(8)\ x'=x''-1,\quad y'=y''-1/b,\\&(9)\ (x'',y'',z')=[(b^2+1)^{1/2}/(2b)](x''',y''',z''').\end{aligned}\tag{9.12}$$

The resulting k is necessarily -1. Hence the hyperbolically symmetric Barnes model can reproduce the open $(k<0)$ FLRW model. It can also reproduce the flat $(k=0)$ FLRW model if the following trick is applied to (9.11) (with a and b being constant). We first transform $x=x'=+B/b$, and then let $b\to 0$. In this way (9.11) becomes $V=ay+B$, and in the limit $a\to 0$ this is the flat FLRW metric.

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APPENDIX A: TRANSFORMING THE HYPERBOLIC TWO-METRIC

We shall transform the two-dimensional metric

$$ds_2^2=d\theta^2+\sinh^2\theta\,d\phi^2\tag{A1}$$

into

$$ds_2^2=d\tau^2+e^{2\tau}dz^2\tag{A2}$$

(compare Refs. 26 and 27). The following formulas alternately present the transformation and its result on the previous metric form:

$$\theta=\ln(1+2p)-\ln(1-2p),\tag{A3}$$

$$ds_2^2=16(1-4p^2)^{-2}(dp^2+p^2\,d\phi^2),\tag{A4}$$

$$u=p\sin\phi-\tfrac{1}{2},\quad v=p\cos\phi,\tag{A5}$$

$$ds_2^2=(u+u^2+v^2)^{-2}(du^2+dv^2),\tag{A6}$$

$$u=[w-(w^2+z^2)]/T,\quad v=z/T,$$

$$T\overset{\text{def}}{=}1-2w+w^2+z^2,\tag{A7}$$

$$ds_2^2=(dw^2+dz^2)/w^2\tag{A8}$$

[if the previous step looks miraculous, then consult Sec. IV, it is the Haantjes transformation in two dimensions, $(u,v)=H(-1,0)(w,z)$],

$$w=e^{-\tau},\tag{A9}$$

and the resulting metric is (A2). ■

APPENDIX B: THE PLANE SYMMETRIC ANALOG OF THE SCHWARZSCHILD–de SITTER SOLUTION

In order to bring out the analogy of (6.2) with Eq. (6.7) in Ref. 42, let us change the variable z and the function w according to

$$z=\ln\zeta,\quad w=R\Phi(t,\zeta)/\zeta.\tag{B1}$$

Then (6.3) will change to

$$\Phi_{,\zeta}=(\Phi/\zeta)\pm[\tfrac{2}{3}C(\Phi/\zeta)^3+F^{-2}(t)-\Lambda/3]^{1/2},\tag{B2}$$

where ζ plays the role of r from Ref. 42, and the metric becomes

$$ds^2=F^2(t)\Phi^{-2}\Phi_{,t}^2\,dt^2-\Phi^{-2}[d\zeta^2+\zeta^2(d\theta^2+d\phi^2)],\tag{B3}$$

where $x=\theta\cos\phi$, $y=\theta\sin\phi$. By the methods of Ref. 42, it can now be shown that a coordinate transformation of the form $t(t',\zeta'),\zeta(t',\zeta')$ exists that preserves the algebraic form of (B3), but changes Φ so that the term F^{-2} in (B2) disappears, thus making the new Φ time independent. The new g_{00} becomes $(\zeta\Phi_{,\zeta}-\Phi)^2$ (primes dropped). The further coordinate transformation $r=\zeta\Phi^{-1}$ reduces then the metric to (6.2) with $K\to 0$ and $m=-C/3$. The generator of the additional symmetry, which in the original coordinates (t,x,y,z) was

$$J_4=-\frac{1}{2F}\left[\frac{(w/R)_{,z}}{(w/R)_{,t}}\right]\frac{\partial}{\partial t}+\frac{1}{2F}\frac{\partial}{\partial z},\tag{B4}$$

is then transformed into $\partial/\partial t$.

APPENDIX C: THE HYPERBOLICALLY SYMMETRIC ANALOG OF THE SCHWARZSCHILD–de SITTER SOLUTION

Let us perform the change of variables inverse to (3.21)–(3.23) in the metric (3.2) with (3.18) and (3.19) and w given by (6.11), i.e.,

$$x=e^{-\tau}\cos(\ln\zeta),\quad y=e^{-\tau}\sin(\ln\zeta),\tag{C1}$$

which implies

$$u=x/y=1/\tan(\ln\zeta),\tag{C2}$$

where ζ plays the role of r from (3.21). The metric then becomes

$$ds^2 = F^2(R/w)^2(w/R)^2_{,t}\,dt^2 - (1+u^2)[R/w(t,u)]^2(d\zeta^2/\zeta^2 + d\tau^2 + e^{2\tau}dz^2). \tag{C3}$$

Let us next introduce the new function $Y(t,u)$ in (6.11) by

$$w = (1+u^2)^{1/2}RY, \tag{C4}$$

then change the variable according to (C2), and finally introduce the new function $\Phi(t,\zeta)$ by

$$Y = \Phi/\zeta. \tag{C5}$$

The resulting equation for $\Phi(t,\zeta)$ will be

$$\Phi_{,\zeta} = \Phi/\zeta \pm [-(\Phi/\zeta)^2 + \tfrac{2}{3}B(\Phi/\zeta)^3 + F^{-2}(t) - \Lambda/3]^{1/2}. \tag{C6}$$

This is analogous to Eq. (6.7) in Ref. 42 [the sign before $(\Phi/\zeta)^2$ is opposite here]. The methods of Ref. 42 work here exactly as described in Appendix B, so a transformation to (6.2) with $K = -1$ exists, where $m = -B/3$ and $r = \zeta'/\Phi$, ζ' being that variable in which $1/F^2$ drops out from (C6). The generator of additional symmetry which in the coordinates of (6.11) is

$$J_4 = \left[(1+u^2)\frac{(w/R)_{,u}}{(w/R)_{,t}} - u\frac{w/R}{(w/R)_{,t}}\right] \times \frac{1}{F}\frac{\partial}{\partial t} - \frac{y}{F}\frac{\partial}{\partial x} + \frac{x}{F}\frac{\partial}{\partial y} \tag{C7}$$

reduces in the coordinates of (6.2) to $J_4 = \partial/\partial t$.

[1] A. Barnes, Gen. Relativ. Gravit. **4**, 105 (1973).
[2] H. Stephani, Commun. Math. Phys. **4**, 137 (1967).
[3] A. Krasiński, Gen. Relativ. Gravit. **13**, 1021 (1981).
[4] A. Krasiński, Gen. Relativ. Gravit. **15**, 673 (1983).
[5] P. Kustaanheimo and B. Qvist, Soc. Sci. Fenni. Comment. Phys. Math. **XIII** (16), 1 (1948).
[6] A. H. Taub, Ann. Inst. H. Poincaré A **9**, 153 (1968).
[7] M. E. Cahill and G. C. McVittie, J. Math. Phys. **11**, 1382 (1970).
[8] E. N. Glass, J. Math. Phys. **20**, 1508 (1979).
[9] A. Banerjee and N. Chakravarty, Acta Phys. Pol. B **10**, 3 (1979).
[10] A. Krasiński, in *Proceedings of the 4th Marcel Grossman Meeting on General Relativity*, edited by R. Ruffini (Elsevier, Amsterdam, 1986), p. 989.
[11] A. Krasiński, in *Gravitational Collapse and Relativity*, Proceedings of Yamada Conference XIV, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986), p. 500.
[12] References 10 and 11 discussed the Barnes models, in particular the plane symmetric one, even though no mention of Barnes's paper was made there.
[13] G. F. R. Ellis, in *General Relativity and Gravitation*, edited by B. Bertotti, F. de Felice, and A. Pascolini (Reidel, Dordrecht, 1984), p. 215.
[14] A. Krasiński, in *The Big Bang and Georges Lemaitre*, edited by A. Berger (Reidel, Dordrecht, 1984), p. 63.
[15] D. A. Szafron, J. Math. Phys. **18**, 1673 (1977).
[16] P. Szekeres, Commun. Math. Phys. **41**, 55 (1975).
[17] G. Lemaitre, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933).
[18] For mysterious reasons, the solution obtained in Ref. 17 is now commonly called "the Tolman–Bondi model" or "the Tolman model" although Tolman (Ref. 19) quoted Lemaitre (Ref. 17) and Bondi (Ref. 20) quoted Tolman (Ref. 19), none of them claiming priority.
[19] R. C. Tolman, Proc. Natl. Acad. Sci. USA **20**, 169 (1934).
[20] H. Bondi, Mon. Not. R. Astron. Soc. **107**, 410 (1947).
[21] M. Oleson, J. Math. Phys. **12**, 666 (1971).
[22] These formulas were calculated by the algebraic program ORTOCARTAN (see also Refs. 23–25).
[23] A. Krasiński, SIGSAM Bulletin **17**, 12 (1983).
[24] A. Krasiński, in *International Conference on Computer Algebra and its Applications in Theoretical Physics*, edited by N. N. Govorun (Joint Institute for Nuclear Research, Dubna, 1986), p. 50.
[25] A. Krasiński and M. Perkowski, *The System ORTOCARTAN—User's Manual* (University of Cologne, Cologne, 1983), 3rd ed. (available as a magnetic tape recording or a printout).
[26] D. Kramer, H. Stephani, E. Herlt, and M. A. H. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge U. P., Cambridge, 1980), pp. 103 and 104.
[27] H. P. Robertson and T. W. Noonan, *Relativity and Cosmology.* (Saunders, Philadelphia, 1968), pp. 190–192.
[28] P. Kustaanheimo, Soc. Sci. Fenn. Comment. Phys Math. **XIII** (12), 1 (1947).
[29] M. Wyman, Phys. Rev. **70**, 396 (1946).
[30] M. Wyman, Aust. J. Phys. **31**, 111 (1978).
[31] G. C. McVittie, Mon. Not. R. Astron. Soc. **93**, 325 (1933).
[32] G. C. McVittie, Ann. Inst. H. Poincaré A **6**, 1 (1967).
[33] N. Chakravarty, S. B. Dutta-Choudhury, and A. Banerjee, Aust. J. Phys. **29**, 113 (1976).
[34] E. N. Glass and B. Mashhoon, Astrophys. J. **205**, 570 (1976).
[35] A. Banerjee and S. Banerji, Acta Phys. Pol. B **7**, 389 (1976).
[36] H. Nariai. Prog. Theor. Phys. **38**, 92 (1967).
[37] M. Wyman, Can. Math. Bull. **19**, 343 (1976).
[38] H. Stephani, J. Phys. A **16**, 3529 (1983).
[39] J. Plebański, "On conformally equivalent Riemannian spaces," Centro de Investigacion y de Estudios Avanzados del I.P.N., Mexico, 1967, p. 326.
[40] J. Haantjes, Koninkl. Ned. Akad. Wet. Proc. **40**, 700 (1937).
[41] J. A. Schouten and J. Haantjes, Koninkl. Ned. Akad. Wet. Proc. **43**, 1288 (1940).
[42] A. Krasiński and J. Plebański, Rep. Math. Phys. **17**, 217 (1980).
[43] C. B. Collins and J. Wainwright, Phys. Rev. D **27**, 1209 (1983).
[44] C. B. Collins, J. Math. Phys. **26**, 2009 (1985).
[45] W. Kinnersley, J. Math. Phys. **10**, 1195 (1969).