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# INHOMOGENEOUS GENERALIZATIONS OF THE ROBERTSON-WALKER COSMOLOGICAL MODELS

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ABSTRACT

Solutions of Einstein s equations are discussed in which the flow of matter is hypersurface-orthogonal and shearfree, but accelerating, the hypersurfaces being conformally flat. The field equations require integrability conditions which are solved. In 4 of the 8 cases the field equations are reduced to a single ordinary differential equation of 2nd order, the other 4 cases are under investigation. In some solutions at an initial instant the matter density is a periodic function of the distance along the curves orthogonal to the orbits of the symmetry group. When the amplitude of the spatial variation of density goes to zero, the flat FLRW model is obtained.

## 1. WHY ARE SUCH GENERALIZATIONS NEEDED?

It is fair to say that no observational evidence of the Universe being homogeneous has been found[1]. Homogeneity is a convenient assumption which makes models simple and is philosophically appealing because of the Copernician principle saying that we do not occupy a preferred position in the Universe. On a small scale (up to groups of galaxies at least) the Universe is evidently inhomogeneous. Thus it may be homogeneous only in the sense that a certain large structure is repetitive (in other words, that the symmetry group defining homogeneity is discrete[1]). It is not known how a discrete symmetry group may be built into a metric. It is clear however that such a spacetime will have no Lie group of symmetries, and so the absence of built-in isometries is a precondition of success.

## 2. HOW CAN THEY BE OBTAINED?

At the beginning, for simplicity, as few assumptions as possible of those underlying the Friedman-Lemaitre-Robertson-Walker (FLRW) models should be rejected. In the first step, it was assumed that homogeneity and isotropy are "intrinsic symmetries" in the sense of Collins[2], i.e. that only spatial sections of the Universe have this symmetry. The resulting Stephani Universe[3-5] turned out not to be general enough: matter-density in it depended only on time. We will therefore make still weaker assumptions:

(A1) The source is the perfect fluid, thus:

$$G_{\alpha\beta} = (8\pi G/c^4)\ (\varepsilon + p)u_\alpha u_\beta - pg_{\alpha\beta}\ , \tag{2.1}$$

(A2) The fluid velocity $u_\alpha$ is irrotational (i.e. orthogonal to a family of spacelike hypersurfaces S).

(A3) The velocity field is shearfree and expanding.

(A4) All hypersurfaces S are conformally flat.

From (A2) it follows that if the flow lines are taken to be the t-coordinate lines, and the spatial coordinates are chosen within the hypersurfaces S, then $g_{oi} = 0 = u_i$, $i = 1, 2, 3$. From (A3) it folows then

$$g_{ij} = h_{ij}\exp\left(-\tfrac{1}{3}\int\theta g_{oo}^{1/2}dt\right) \tag{2.2}$$

where $h_{ij,t} = 0$ and $\theta$ is the expansion scalar. From (A4) it follows further that $h_{ij}$ is proportional to the flat 3-metric, and so, in suitably chosen coordinates,

$$ds^2 = D^2dt^2 - (R^2/V^2)\ (dx^2 + dy^2 + dz^2), \tag{2.3}$$

where $D = D(t,x,y,z)$, $R(t)$ and $V(t,x,y,z)$ are functions to be determined from the field equations.

## 3. THE FIELD EQUATIONS

From $G_{oi} = 0$, $i = 1,2,3$, we conclude that either

$$(V/R)_{,t} = 0 \tag{3.1}$$

or

$$D = F(t)\ (R/V)\ \frac{\partial}{\partial t}\ (V/R), \tag{3.2}$$

where $F(t)$ is an arbitrary function. The case (3.1) is not interesting for cosmology because it implies $\dot{\varepsilon} = 0 = \theta$. Therefore we will follow only (3.2) (which is a restatement of (2.1) since $\theta = -3/F$). Then, from $G_{ij} = 0$, $i \neq j$:

$$R\ V_{,ij}\ /\ V^2 = F_k(x,\ y,\ z)\ , \tag{3.3}$$

where $(i,j,k) = (1,2,3)$ cyclically and $F_k$ are arbitrary functions. Further, from $G_{ii} - G_{jj} = 0$ (no summation)

$$R\ (V_{,ii} - V_{,jj})/\ V^2 = G_k(x,\ y,\ z), \tag{3.4}$$

with $G_k$ being other arbitrary functions. With $F_k = G_k = 0$ and $V_{,t} \neq 0$ the Stephani solution follows, if further $V_{,t} = 0$, the FLRW solutions result.

Eqs. (2.1) are now solved. However, (3.3) - (3.4) are 5 equations (because $G_2 \equiv -G_1 - G_3$) for the function $V$, so integrability conditions are necessary. They have the form:

$$M^i{}_A V_{,i}/\ V = W_A, \tag{3.5}$$

where $A = 1,\ldots,5$; the $3{\times}5$ matrix $M$ and the 5-vector $W$ being determined by $F_k$, $G_k$ and their derivatives. At most 3 equations in (3.5) can be independent, otherwise the set cannot be algebraically solved for $V_{,i}/V$. However, with 3 independent equations, one of the following cases occurs:

1. Either $V_{,t} = 0$ and a FLRW model (not even Stephani!) results, which is nothing new;

2. Or $V\ (t,x,y,z) = R(t)\ H(x,y,z)$, and the stationary case (3.1) results which we left out of consideration.

A new solution can thus result only if at most 2 equations in (3.5) are independent. This means:

Every $3\times 3$ sub-matrix of M is singular. (3.6)

Each 3 equations in (3.5) are linearly dependent. (3.7)

Two cases must be considered separately now:

1. The degenerate case when $F_1\ F_2\ F_3 = 0$,
2. The generic case when $F_1\ F_2\ F_3 \neq 0$.

## 4. THE SOLUTIONS FOR THE DEGENERATE CASE

Eqs (2.1) reduce here to one of the following cases:

Case I: The plane-symmetric Universe:

$$R\ (t)\ V,_{zz}/V^2 = f_P(z), \tag{4.1}$$

where $f_P(z)$ is an arbitrary function. Only the flat FLRW model is contained here which results with $f_P = 0$, $V = 1$.

Case II: The line-homogeneous Universe:

$$R\ (t)\ w,_{gg}/w^2 = f_L(g), \tag{4.2}$$

where

$$g = x/y, \quad w\ (t,g) = V/y, \tag{4.3}$$

and $f_L(g)$ is an arbitrary function. These solutions generalize also only the flat FLRW model. Note that (4.1) and (4.2) is the same equation, only the meaning of the variable and of the function is in each case different.

## 5. THE SOLUTIONS FOR THE GENERIC CASE, $F_1\ F_2\ F_3 \neq 0$

The equations resulting from (3.6) are solved here by:

$$G_1 = F_1(F_3/\ F_2 - F_2/\ F_3)\ ,$$
$$G_3 = F_3(F_2/\ F_1 - F_1/\ F_2)\ , \tag{5.1}$$

while those resulting from (3.7), with the substitutions:

$$F_2 = P_2F_1\ , \qquad F_3 = P_3F_1 \tag{5.2}$$

reduce to the following set:

$$- P_2,_x - P_2,_y\ /\ P_2 + P_3\ (P_2 - 1/P_2),_z = 0, \tag{5.3}$$

$$- P_3,_x - P_3,_z\ /\ P_3 + P_2\ (P_3 - 1/P_3),_y = 0, \tag{5.4}$$

$$- P_3,_y + P_3P_2,_y/P_2 + P_2,_z - P_2P_3,_z/P_3 + (P_3/P_2 - P_2/P_3),_x = 0\ . \tag{5.5}$$

Note that $F_1$ does not appear here. The solutions of (5.3) -

(5.5) must then be substituted into the integrability conditions (3.5) (of which only two remain in virtue of (3.6) - (3.7)), and after (3.5) are solved, the solution of (3.3) - (3.4) is guaranteed to exist, but must be found in the next step. Several cases have to be considered separately:

<u>Case IIa:</u> $P_2$ = const fulfills (5.3). Then $P_3$ is found from (5.4). The subcase $P_3$ = const leads to (4.1), but in coordinates rotated and dilatated with respect to those of (4.1). If $P_3$ is not constant, then (4.2) results, again in rotated and dilatated coordinates.

<u>Case III:</u> The spherically symmetric Universe:
If $P_{2,z} = 0$, then (5.3) is independent of the other equations and can be solved. If in addition $P_{3,y} = 0$, then (5.4) can also be solved independently. In this case, the solution of (5.3) - (5.5) is

$$P_2 = x/y, \quad P_3 = x/z \tag{5.6}$$

and (3.3) - (3.4) reduce to the following equation:

$$R(t)\, V_{,uu} / V^2 = f_S(u) , \tag{5.7}$$

where

$$u = x^2 + y^2 + z^2 , \tag{5.8}$$

and $f_S(u)$ is an arbitrary function. The subcases of (5.7) include the spherically symmetric Stephani solution (which results with $f_S = 0$, $V_{,t} \neq 0$) and <u>all</u> the FLRW solutions (if in addition $V_{,t} = 0$). Note that (5.7) is again the same as (4.1), with a still different meaning of the variable.

<u>Case IV:</u> The axially symmetric Universe:
If $P_{2,z} = 0 \neq P_{3,y}$, then the solution of (5.3) - (5.5) is

$$P_2 = x/y, \quad P_3 = x/(z + L), \tag{5.9}$$

where L (x,y,z) is given by

$$x^2 + y^2 + (z + L)^2 = Q^2(L), \tag{5.10}$$

Q(L) being an arbitrary function. The case of constant L is singular, then (5.10) does not hold and Case III is reobtained. The field equations (3.3) - (3.4) reduce here to

$$R\ (t)\ W_{,ZZ} / W^2 = f_A(Z), \tag{5.11}$$

where

$$Z = (C - x^2 - y^2 - z^2)/(2z), \quad W = V/z, \; C = \text{const}, \tag{5.12}$$

and $f_A(Z)$ is an arbitrary function. With $f_A = 0 \neq W_{,t}$, the axially symmetric case of the Stephani solution results, if further $W_{,t} = 0$, then all the 3 FLRW solutions are recovered. Note that (5.11) is (4.1) in one more disguise.

When $P_{2,z} \neq 0 \neq P_{3,y}$, the complete set of solutions of (5.3) - (5.5) was found, but the corresponding solutions of (3.3) - (3.4) are not yet known (work in progress).

Case V: The most general solution of (5.3) - (5.5) is

$$P_2 = A[x - H(A)]/[Ay + F(A)], \tag{5.13}$$

$$P_3 = [x - H(A)]/[z + L(A)], \tag{5.14}$$

where $H(A)$, $F(A)$ and $L(A)$ are arbitrary functions of $A(x,y,z)$ which is given implicitly by

$$A^2(x - H)^2 + (Ay + F)^2 + A^2(z + L)^2 = 1. \tag{5.15}$$

This solution is obtained under the assumption that $A$ is not constant. It reduces to Case IV when $H = F = 0$ (with $A = 1/Q$) and to Case III if $L = 0$ in addition.

Case VI: If $A$ = const, then

$$P_2 = A[x - H(F)]/(Ay + F), \tag{5.16}$$

$$P_3 = [x - H(F)]/[z + L(F)], \tag{5.17}$$

where now $H(F)$ and $L(F)$ are arbitrary and $F(x,y,z)$ is given by (5.12) with $A$ = const. Here $F$ cannot be constant.

Case VII: If $A$ and $F$ are constant and $A \neq 0$, then

$$P_2 = A(x - H)/(Ay + F), \tag{5.18}$$

$$P_3 = (x - H)/[z + L(H)], \tag{5.19}$$

where $L(H)$ is arbitrary and $H(x,y,z)$ is given by (5.12) with $A$ and $F$ being constant.

Case VIII: If $A \longrightarrow 0$, then

$$P_3 = - P_2[z + L(P_2)]/(xP_2 + y), \tag{5.20}$$

where $L(P_2)$ is arbitrary and $P_2(x,y,z)$ is given by

$$F(P_2)(xP_2 + y) = - (z + L)[1 - F^2(P_2) - P_2^2 F^2(P_2)]^{1/2}, \tag{5.21}$$

$F(P_2)$ being another arbitrary function.

The proof that Cases IIa - VIII are all solutions of (5.3) - (5.5) is contained in their derivation which will be published separately.

## 6. AN EXAMPLE OF A SOLUTION OF (4.1)

In any solution of (4.1) the pattern of inhomogeneity will be one-dimensional. However, if the matter distribution is a periodic function of the distance in the direction of z, then we can expect it to be a first approximation to a model with 3-dimensional spatial periodicity.

Let us rewrite (4.1) in terms of the variable l where

$$l(t,z) = R(t) \int dz/V(t,z), \tag{6.1}$$

i.e. l is the geodesic distance in the z-direction. Then

$$R^3(t)V,_{ll}/V^4 - R^3(t)V,_l{}^2/V^5 = f_P(z(l)). \tag{6.2}$$

Note that if V is a periodic function of l, then so will be $V,_z$. Consequently, also $\rho$ will be periodic in l, since

$$8\pi G\rho/c^2 = 2f_P V^3/R^3 - 3V,_z{}^2/R^2 + 3/F^2. \tag{6.3}$$

V can be periodic in l at an initial instant $t = t_o$(*). We first note that (6.2) can be understood as the definition of $f_P(z(l))$: we choose $V(t_o,l)$ arbitrarily, e.g.:

$$V(t_o,l) = A + B \sin(Cl), \tag{6.4}$$

where A, B and C are constants and $A > |B|$. Then

$$f_P(l) = -R^3BC^2[A + B\sin(Cl)]^{-5}[B + A\sin(Cl)] \tag{6.5}$$

$$8\pi G\rho/c^2 = -BC^2[A + B\sin(Cl)]^{-2}[2B + 2A\sin(Cl) + 3B\cos^2(Cl)] + 3/F^2 \tag{6.6}$$

and $\rho(t_o,l)$ is indeed periodic in l. The function $f_P(l)$ can be reexpressed in terms of z with use of (6.1), i.e.:

$$R(t_o)z = Al + (B/C)[1 - \cos(Cl)]/R(t_o). \tag{6.7}$$

The equation (6.2) will then determine V(t,l) at other times. The solution of (6.2) will contain two arbitrary functions of t, so the evolution of V will be indeterminate. This is because we did not consider an equation of

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(*) In a previous version of this text[6] V and $\rho$ were constructed as periodic functions of the coordinate z at $t = t_o$. Since z has no invariant meaning, that construction did not yield a spacetime with a discrete symmetry group which was its aim. I thank Dr. J. Gruszczak for this remark.

state. The evolution of $V$ can be qualitatively discussed if the energy conditions of Hawking and Ellis[7] are imposed. The periodicity of $\rho$ will persist in time if also $V,_t$ is periodic in $l$ at $t = t_o$ (this corresponds to the velocity field of matter being periodic in $l$ at $t = t_o$). Otherwise, the structure will dissolve during evolution into an irregular pattern.

With $B = 0$, eq. (6.4) gives the flat FLRW model. Note that a model with constant $f_p$ in which (4.1) can be integrated to a first order equation is not interesting since then $\rho,_z = 0$ and so $\rho = \rho(t)$.

## 7. COMMENTS

The previous section shows that initial conditions can be set up so that the matter-density will be periodic in one spatial direction at $t = t_o$. What happens with this structure as time proceeds, remains to be investigated. The structure is not complicated enough to be a model of the real Universe (but is more general than the flat FLRW model!), nevertheless, can be useful as an example for studying the consequences of averaging physical quantities over space. In the terminology of Ellis[1] the model of sec. 6 can be said to be at scale $4\frac{2}{3}$, intermediate between scale 5 (the FLRW models) and scale 4 appropriate for describing the evolution of superclusters of galaxies; the structure being smoothed out in 2 of the 3 spatial dimensions.

In cases II, III and IV (eqs. (4.2), (5.7) and (5.11)) it is difficult to verify whether $\rho$ can be a periodic function of the invariant distance $l$ since $V$ would have to contain nonperiodic terms as well, to compensate for nonperiodic factors in $\rho$. Even if $\rho$ is periodic in $l$, however, owing to the symmetries present in those cases, the distribution of extrema of $\rho$ will have nothing to to with large-scale homogeneity, and so these other models are just totally inhomogeneous.

The solutions discussed here are still very special: each of them depends on the spatial coordinates through a single variable, and so they represent one-dimensional structures. This feature results most probably from the assumption of spatial conformal flatness which is very strong and artificial from the physical point of view. Thus it remains as a challenge for the future to obtain models with 3-dimensional patterns of inhomogeneity.

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An earlier stage of this work was described in the Proceedings of the 4th Marcel Grossman Meeting in Rome (reference below). Part of that material was used here with the permission of the North-Holland Publishing Company.

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