PHYSICAL PROPERTIES OF THE EXTENDED CHASLES EQUILIBRIUM FIGURE

Stanisław L. BAŻAŃSKI [1]
Institute of Theoretical Physics, Warsaw University, Hoża 69, 00 681 Warsaw, Poland

Renata KACZYŃSKA
Nowolipie 29/12, 01 002 Warsaw, Poland

and

Andrzej KRASIŃSKI
N. Copernicus Astronomical Center, Polish Academy of Sciences, Bartycka 18, 00 716 Warsaw, Poland

Received 18 September 1985; revised manuscript received 18 January 1986; accepted for publication 18 January 1986

Hydrodynamical functions for an object whose newtonian gravitational field is constant on confocal ellipsoids of revolution are investigated. The conditions: pressure $\geqslant 0$ and (angular velocity field)$^2 \geqslant 0$ are fulfilled, but a ring singularity in matter density or a disk singularity in the angular velocity distribution are inevitable.

1. Introduction. A solution of the Poisson equation and the Euler equations of motion found in 1980 [1] describes the interior and exterior gravitational fields having oblate confocal ellipsoids as equipotential surfaces. However, the functions describing the fluid source of that field were not all given by closed-form formulae: the pressure p was determined by a linear partial differential equation (see next section), and the angular velocity ω was given as an expression for ω^2 which involved derivatives of p (not known explicitly). In this paper we present the solution in a form in which the conditions p(outer surface) = 0 and p(inside the source) $\geqslant 0$ are fulfilled evidently, and necessarily imply $\omega^2 \geqslant 0$. This form enables one to conclude that a singularity either in the mass density ρ on the focal ring of the ellipsoids or in ω on the central disk is unavoidable. The exterior field discussed here was first found by Chasles in 1840 [2]. However, the source given by Chasles (a layer of mass of finite surface density) was rather artificial. It was replaced by a continuous spatial distribution of perfect fluid in a previous paper [1]. For this reason we call the configuration "the extended Chasles equilibrium figure".

2. Equations defining the source. We list without repeating the proofs the results of a previous paper [1]. The oblate spheroidal coordinates (r, θ, ϕ) used here are defined in terms of the cartesian coordinates (x, y, z) by

$$x = D \sin\theta \cos\phi \,, \quad y = D \sin\theta \sin\phi \,,$$

$$z = r \cos\theta \,, \quad D := (r^2 + a^2)^{1/2} \,. \tag{2.1}$$

The surfaces r = const. are confocal oblate ellipsoids of revolution, the surfaces θ = const. are one-sheet hyperboloids of the same focal ring $\{(x^2 + y^2)^{1/2} = a,\ z = 0\}$ [which is the locus of singularity of the coordinates (r, θ, ϕ)]. The mass density inside the source is

$$\rho(r, \theta) = f(r)/J \,, \quad J := r^2 + a^2 \cos^2\theta \,, \tag{2.2}$$

where $f(r)$ is an arbitrary function. The total mass of the source is $M = M(R)$ where

[1] Partly supported by the Polish Research Project MR I-7.

$$M(R) = 4\pi \int_0^R f(r)\,\mathrm{d}r\,, \tag{2.3}$$

$r = R$ being the outer surface of the source. The exterior gravitational potential in the point (r, θ, ϕ) is given by

$$V_e(r) = -(G/a)M(R)\arctan(a/r)\,. \tag{2.4}$$

The interior potential depends also only on r and is

$$V_i(r) = \int_0^r [4\pi G/(r'^2 + a^2)]\,\mathrm{d}r' \int_0^{r'} f(r'')\,\mathrm{d}r'' + V_0\,, \tag{2.5}$$

where V_0 is a constant whose value ensures that $V_i(R) = V_e(R)$. The field of pressure is determined by the equation

$$[D^2/f(r)]\cos\theta\, p_{,r} - [r\sin\theta/f(r)]p_{,\theta} + G\cos\theta\, M(r)/J = 0\,, \tag{2.6}$$

and the field of angular velocity is

$$\omega^2(r,\theta) = [r/f(r)]p_{,r} + [\cot\theta/f(r)]p_{,\theta} + GrM(r)/(D^2J)\,. \tag{2.7}$$

3. *The solution for pressure*. Eq. (2.6) is solved by the standard method of solving linear partial differential equations of the first order (see e.g. ref. [3] for details). The general solution $p(r, \theta)$ of (2.6) is determined by

$$F(\psi_1(p,r,\theta), \psi_2(p,r,\theta)) = 0\,, \tag{3.1}$$

where F is an arbitrary function, while ψ_1 and ψ_2 are functions such that $\psi_1 = C_1 =$ const. and $\psi_2 = C_2 =$ const. are the first integrals of the following set of ordinary differential equations implied by (2.6):

$$\frac{f(r)\,\mathrm{d}r}{D^2\cos\theta} = -\frac{f(r)\,\mathrm{d}\theta}{r\sin\theta} = -\frac{J\,\mathrm{d}p}{G\cos\theta\, M(r)}\,. \tag{3.2}$$

The first integrals of (3.2) are

$$\psi_1(p,r,\theta) := D\sin\theta = C_1\,, \tag{3.3}$$

$$\psi_2(p,r,\theta) := p + \Phi(r, D\sin\theta) = C_2\,, \tag{3.4}$$

where

$$\Phi(x,y) := G\int \frac{f(x)M(x)}{(x^2+a^2)^2 - a^2y^2}\,\mathrm{d}x\,. \tag{3.5}$$

After (3.4) and (3.5) are substituted into (3.1), the resulting equation can be solved for p:

$$p(r,\theta) = -\Phi(r, D\sin\theta) + \widetilde{H}(D\sin\theta)\,, \tag{3.6}$$

where $\widetilde{H}$ is an arbitrary function [it fulfills the homogeneous part of (2.6)]. This can be written in an equivalent form which is more useful in calculations:

$$p(r,\theta) = -G\int_R^r f(x)M(x)W^{-1}(r,x,\theta)\,\mathrm{d}x + H(D\sin\theta)\,, \tag{3.7}$$

where

$$W(r,x,\theta) := (x^2+a^2)^2 - a^2\sin^2\theta\, D^2\,, \tag{3.8}$$

$$H := \widetilde{H} - \Phi(R, D\sin\theta)\,, \tag{3.9}$$

and R is an arbitrary constant. We shall choose R so that $r = R$ is the outer surface of the source. If we now impose the free surface boundary condition $p(R,\theta) = 0$, then $H = 0$. In that case, $\rho \geqslant 0$ implies $p(r,\theta) \geqslant 0$: from (2.2) we have $f(r) \geqslant 0$, so $M(r) \geqslant 0$ from (2.3); furthermore $W(r,x,\theta) \geqslant D^2J \geqslant 0$ because $r \leqslant x \leqslant R$ in the integration range and so $p \geqslant 0$ for $r \leqslant R$ in (3.7).

4. *The angular velocity and singularities*. We find from (2.7) with the help of (2.3),

$$\omega^2 = -[Ga^2J/2\pi f(r)]\int_R^r M(x)(\mathrm{d}M/\mathrm{d}x)W^{-2}(r,x,\theta)\,\mathrm{d}x + JH'/[f(r)D\sin\theta]\,, \tag{4.1}$$

where H' is the derivative of H with respect to its argument.

With $H = 0$ we have $\omega^2 \geqslant 0$ necessarily, by the reasoning used after (3.9), and moreover $\omega(R,\theta) \equiv 0$. We have thus shown that the solution considered here is not unrealistic. We recall however that the motion in the source is shearing unless $\omega = a = 0$ [1]. Thus the fluid must be nonviscous or else the flow cannot be stationary. This is not what one expects from a real star. In addition, a singularity is necessarily present on the disk $r = 0$ or on its edge $\{r = 0, \theta = \pi/2\}$. This can be seen from (2.2), (3.7) and (4.1) if we consider the following two cases:

Case A.

$$\mathrm{d}M/\mathrm{d}r = 4\pi f(r) \underset{r\to 0}{\longrightarrow} 0 \quad \text{so that} \lim_{r\to 0} [f(r)/r^2] < \infty \,. \tag{4.2}$$

In this case, $\rho(r,\theta)$ and $p(r,\theta)$ are finite everywhere (but the values of $\rho(0,\pi/2)$ and $p(0,\pi/2)$ will in general depend on the path of approaching the ring $\{r=0, \theta=\pi/2\}$). However, for $\theta \neq \pi/2$, $\omega \to \infty\,(r\to 0)$. This "explains" why $\rho = 0$ at $r=0, \theta \neq \pi/2$ in this case: all matter inside the disk is swept out by an infinitely fast rotation.

Case B.

$$f(0) > 0 \,. \tag{4.3}$$

Then, with $H=0$, $\omega^2(0,\theta)$ will be finite for $\theta \neq \pi/2$. However, $\rho(r,\theta)$, $p(r,\theta)$ and $\omega^2(r,\theta)$ will all be singular on the ring $\{r=0, \theta=\pi/2\}$.

There are other cases possible, but, with $H=0$, in every other case *both* the ring-singularity in ρ and p and the disk-singularity in ω^2 will appear. The singularities are invisible from outside and thus physically harmless because it is seen from (2.3) that $M(0)=0$.

Because of the property $\omega(R,\theta)=0$, a distant star, if described by this model, would not reveal its rotation: there would be no Doppler-broadening of spectral lines.

5. No way to avoid a singularity. Since the exterior potential (2.4) has a finite limit for $r\to 0$, one could avoid the singularities by leaving a vacuum cavity inside the source so that the ring $\{r=0, \theta=\pi/2\}$ is contained in the cavity. This is not a physical situation, however.

Another conceivable way to avoid the singularities would be to give up the boundary condition $p(R)=0$ and replace it with the condition of regularity at $r=0$. This simply does not work. The functions ρ and p are finite at $r=0$ only in case A. In order to prevent then ω^2 from diverging at $\{r=0, \theta\neq\pi/2\}$ one must adjust H in (4.1) so that

$$H'(a\sin\theta)/(a\sin\theta)$$

$$= -(Ga^2/2\pi)\int_0^R M(x)(\mathrm{d}M/\mathrm{d}x)W^{-2}(0,x,\theta)\,\mathrm{d}x \,. \tag{5.1}$$

This can be integrated with the result

$$H(D\sin\theta)$$

$$= -(G/4\pi)\int_0^R M(x)(\mathrm{d}M/\mathrm{d}x)W^{-1}(r,x,\theta)\,\mathrm{d}x \,. \tag{5.2}$$

The result of substituting such H in (3.7) and (4.1) is the same as if $H=R=0$. Then, however, $W(r,x,\theta)$ will have a zero at a certain x in the integration range for each $r \geqslant 0$, and so both p and ω^2 will be singular on a two-dimensional surface while p will even cease to be positive-definite.

Still another conceivable way to remove the singularity would be to replace the interior of a certain surface $g(r,\theta)=0$ with a different distribution of matter. The continuity of the potential can be achieved e.g. by inserting a Maclaurin spheroid inside the surface $r=r_1<R$. Then, however, pressure, density and angular velocity will suffer discontinuities at $r=r_1$. It seems certain that problems with continuity will arise with any other distribution of matter, but we leave this question as a challenge for the future.

These difficulties are an example of what can go wrong inside a source even if it matches smoothly to a given exterior field. This danger thus exists for the (still unknown explicitly) perfect fluid source of the Kerr metric [4,5], although Roos [6] has shown that the field equations are integrable in the vicinity of the outer surface.

6. The equation of state and temperature distribution. Apart from the limiting case $a=0$, the body considered here cannot obey the simple equation of state $p=F(\rho)$. The equation of state must explcitly involve a third function, e.g. temperature. One possibility is the ideal gas equation:

$$T = Cp/\rho \,, \tag{6.1}$$

where $C=$ const. Using (2.2) and (3.7) with $H=0$ one concludes easily the following:

(i) In case A, the temperature has a singularity inside the disk $r=0$. On the ring $\{r=0, \theta=\pi/2\}$ the temperature may have a finite or infinite limit depending on the path on which the ring is approached.

(ii) In case B the temperature has no singularity.

On every surface $r=$ const. $<R$ the temperature has a minimum on the symmetry axis ($\theta=0$ and $\theta=$

π) and a maximum on the equator ($\theta = \pi/2$). Note that $T(R) = 0$, this equation of state is thus not very realistic.

7. *The distribution of pressure*. From (3.7) one can find in case A that $p_{,r}(0, \theta) = 0$ and $p_{,rr}(0, \theta) > 0$. Thus p has a local minimum at $r = 0$ for every value of θ, including $\theta = \pi/2$. Since $p \geqslant 0$ and $p(R) = 0$, it follows that in this case p must have (at least one) local maximum somewhere between $r = 0$ and $r = R$ for every π. In case B, $p_{,r}(0, \theta) = 0$ for $\theta \neq \pi/2$, but the sign of $p_{,rr}(0, \theta)$ cannot be determined (it depends on the shape of $f(r)$ in the neighborhood of $r = 0$). Thus in case B, p always has an extremum at $r = 0, \theta \neq \pi/2$, but it is either maximum or minimum. At $r = 0, \theta = \pi/2$, both $p_{,r}$ and $p_{,rr}$ are singular in case B.

In both cases one sees easily from (3.7) that for every r, $0 < r < R$, p has minima at $\theta = 0$ and $\theta = \pi$ (on the symmetry axis) and a maximum on the equator $\theta = \pi/2$.

The distribution of density was discussed in ref. [1]. The distribution of ω^2 cannot be discussed without specifying $f(r)$. For every r, ω^2 has extrema at $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$, but which of them is minimum and which is maximum depends on the shape of $f(r)$.

8. *A symmetry of the gravitational field*. Let us consider the following transformations of the space:

(i) Let each point move within its r = const. surface, parallel to the plane $x = 0$ and counterclockwise so that:

(a) each point in the plane $x = 0$ has its initial coordinate θ changed by $\Delta\theta$, the same for all points;

(b) points in a plane $x = \text{const.} \neq 0$ (which move on an ellipse similar to the ellipse $\{x = 0, r = \text{const.}\}$) are displaced between positions corresponding to those in (a) under the similarity transformation.

(ii) An analogous motion parallel to the plane $y = 0$.

(iii) A rotation around the z-axis.

These transformations reduce to rotations around the x, y and z axes in the limit $a \to 0$, i.e. when the ellipsoids degenerate into spheres. With $a \neq 0$, they are not isometries of the space, but are symmetries of the potential. Their generator, calculated from:

$$J_i = \lim_{\Delta\theta \to 0} \mathrm{d}x_i/\mathrm{d}(\Delta\theta) , \tag{8.1}$$

are, respectively:

$$J_{yz} = \sin\phi \; \partial/\partial\theta + \cos\phi \cot\theta \; \partial/\partial\phi , \tag{8.2}$$

$$J_{xz} = \cos\phi \; \partial/\partial\theta - \sin\phi \cot\theta \; \partial/\partial\theta , \tag{8.3}$$

and $J_{xy} = \partial/\partial\phi$. Thye have thus, in the spheroidal coordinates, the same form as the generators of rotations have in spherical coordinates and form the algebra of the O(3) group. This is an analogy to the problem of collineations of the Riemann tensor (which describes the gravitational field in Einstein's theory) which are not symmetries of the spacetime. See refs. [7–9] for more details.

References

[1] A. Krasiński, Phys. Lett. 80A (1980) 238.
[2] S. Chandrasekhar, Ellipsoidal figures of equilibrium (Yale University Press, New Haven, 1969) p. 46.
[3] E. Goursat, Cours d'analyse mathématique, Tome II (Gauthier-Villars, Paris, 1949) pp. 586–590.
[4] A. Krasiński, Ann. Phys. 112 (1978) 22.
[5] R.P. Kerr, Phys. Rev. Lett. 11 (1963) 237.
[6] W. Roos, Gen. Rel. Grav. 7 (1976) 431.
[7] A. Krasiński, in: 10th Intern. Conf. on General relativity and gravitation (University of Padua, 1983) p. 290.
[8] G.H. Katzin, J. Levine and W.R. Davis, J. Math. Phys. 10 (1969) 617.
[9] C.B.G. McIntosh et al., J. Phys. A13 (1980) 2995; A14 (1981) 2331; J. Math. Phys. 23 (1982) 436.