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A SPATIALLY PERIODIC GENERALIZATION OF THE FLRW COSMOLOGICAL MODELS

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A class of exact inhomogeneous solutions of the Einstein field equations is discussed which contains the Friedman-Lemaitre-Robertson-Walker (FLRW) metrics as a limitting case. The class is defined by the flow of matter being hypersurface-orthogonal and shearfree, but accelerating, while the hypersurfaces are conformally flat. The resulting class of solutions is shown to contain such Universes in which at a certain initial instant the matter density is a periodic function of one spatial variable. In the limit when the amplitude of the spatial variation of density goes to zero, the FLRW models are obtained. This shows that discussing structure-development in the Universe within the exact theory of gravitation is not totally hopeless.

1. WHY?

It is fair to say that no observational evidence of the Universe being homogeneous has been found[1], in spite of a lot of propaganda to the contrary. Homogeneity is just a convenient assumption which makes models simple and is philosophically appealing because of the Copernician principle saying that we do not occupy a preferred position in the Universe. On a small scale (up to groups of galaxies at least) the Universe is evidently very inhomogeneous. Thus it may be homogeneous only in the sense that a certain structure, larger than a group of galaxies, is <u>repetitive,</u> i.e. that the matter distribution is <u>spatially periodic</u> (or, in other words, that the symmetry group defining homogeneity is discrete[1]). Such a Universe is automatically homogeneous on the average if physical quantities are averaged over the volume of the repetitive structure. There are serious problems of principle with averaging, e.g. as conjectured by Ellis[1] and proven by Carfora and Marzuoli[2], if Einstein's equations are fulfilled on the fine scale before averaging, then they will not hold after averaging: new source terms will show up which can seriously influence the dynamics of geometry and matter. However, it is expected that an averaged model will have a FLRW geometry so that the generalized model can inherit all the successes of the classical models.

2. HOW?

For the beginning, as few assumptions as possible of those underlying the FLRW models should be rejected. In the first step, it was assumed that homogeneity and isotropy are "intrinsic symmetries" in the sense of Collins[3], i.e. that only the spatial sections of the Universe have this symmetry while the whole spacetime has no symmetry a priori. The resulting Stephani Universe[4-6] turned out not to be general enough: matter-density in it depended only on time. We will therefore make the still weaker assumption that the hypersurfaces orthogonal to the fluid flow are conformally flat. More precisely, we assume the metric form to be:

$$ds^2 = D^2dt^2 - (R^2/V^2)(dx^2 + dy^2 + dz^2), \tag{2.1}$$

where $D = D(t,x,y,z)$, $R(t)$ and $V(t,x,y,z)$ are fuctions to be determined from the field equations, and t is a parameter on the flow-lines of matter.

There are differences between (2.1) and the Szekeres-type models[7-12] which have also conformally flat slices. In the latter, it is only required that the Ricci 3-tensor $R_{AB}(t,x^A)$ $(A, B = 1, 2, 3)$ fulfills the equations

$$-R_{AB;C} + R_{AC;B} + (1/4)(g_{AB} R,_C - g_{AC} R,_B) = 0$$

and has one double eigenvalue. The resulting 3-metric for the space $t = \text{const}$ is more complicated, it can be brought to the form $a^2(dx^2 + dy^2 + dz^2)$ in each spacelike slice separately, but not in all the slices simultaneously, unless time-space terms in the 4-metric are allowed. Thus (2.1) puts a stronger limitation on the 3-geometries of the slices. On the other hand, matter in the Szekeres-type models moves on geodesics what happens in (2.1) only with $D = D(t)$, and so (2.1) is more general in this respect. In fact, the only common subset of (2.1) and the Szekeres-type metrics are the FLRW solutions.

The source term in the Einstein's equations will be the perfect fluid, thus:

$$G_{\alpha\beta} = (8\pi G/c^4)[(\varepsilon + p)u_\alpha u_\beta - pg_{\alpha\beta}], \qquad u_\alpha = D\delta^0{}_\alpha. \tag{2.2}$$

No definite equation of state will be assumed.

3. THE FIELD EQUATIONS

From the equations $G_{oi} = 0$, $i = 1,2,3$, we conclude that either

$$(V/R),_t = 0 \tag{3.1}$$

or

$$D = F(t)\ (R/V)\ \frac{\partial}{\partial t}\ (V/R), \tag{3.2}$$

where $F(t)$ is an arbitrary function. The case (3.1) is not interesting for cosmology because it implies that matter-density is time-independent and the scalar of expansion is zero. Therefore we will follow only (3.2). Then, from the 3 equations $G_{ij} = 0$, $i \neq j$; $i,j = 1, 2, 3$ we obtain:

$$R\ V,_{ij}\ /\ V^2 = F_k(x,\ y,\ z)\ , \tag{3.3}$$

where $(i,j,k) = (1,2,3)$ cyclically and F_k are arbitrary functions of the space-coordinates. Further, from $G_{ii} - G_{jj} = 0$ (no summation, $i,j = 1,2,3$) we obtain

$$R\ (V,_{ii} - V,_{jj})/\ V^2 = G_k(x,\ y,\ z), \tag{3.4}$$

(no summation), with i, j, k as before and G_k being other arbitrary functions. With $F_k = G_k = 0$ and $V,_{ii} \neq 0$ the Stephani solution follows, if further $V, = 0$, the FLRW solutions result. Thus the set of solutions of (3.3) - (3.4) is not empty.

In fact, eqs. (2.2) are already solved. However, (3.3) and (3.4) are 5 equations (because $G_2 \equiv -G_1 - G_3$) for the function V, and so integrability conditions are necessary. They have the form:

$$M^i{}_A V,_i/\ V = W_A, \tag{3.5}$$

where $i = 1, 2, 3$; $A = 1,\ldots,5$; the 3×5 matrix M and the 5-vector W being determined by F_k, G_k and their derivatives, thus M and W do not depend on t. At most 3 equations in the set (3.5) can be independent, otherwise the set cannot be algebraically solved for

$V_{,i}/V$. However, an elementary analysis of (3.5) shows that with exactly 3 equations being independent, one of the following two cases must occur:

1. Either $V_{,ij} = 0$ and a FLRW model (not even Stephani!) results, which is nothing new;

2. Or $V\ (t,x,y,z) = R(t)\ H(x,y,z)$, and the stationary case (3.1) results. We left it out of consideration anyway, but with (3.2), on which all further equations are based, this would lead to the singular result $D = 0$.

We can thus expect a new cosmological model only when at most 2 equations in the set (3.5) are independent. This is equivalent to 2 requirements:

Every 3×3 sub-matrix of M is singular. (3.6)

Each subset of 3 equations in (3.5) is linearly dependent. (3.7)

In order to translate these statements into equations, two separate cases must be considered:

1. The degenerate case when $F_1 \cdot F_2 \cdot F_3 = 0$,
2. The generic case when $F_1 \cdot F_2 \cdot F_3 \neq 0$.

4. THE SOLUTIONS FOR THE DEGENERATE CASE

The integrability conditions (3.5) are completely solved in this case. The field equations (2.2) reduce to one of the following two ordinary differential equations:

4.1. The plane-symmetric Universe:

$$R\ (t)\ V_{,zz}/V^2 = f_P(z), \tag{4.1}$$

where z is a coordinate, $V = V\ (t,z)$, and $f_P(z)$ is an arbitrary function. Since $V_{,x} = V_{,y} = 0$, these solutions contain only the flat FLRW model which results wiht $f_P = 0$, $V = 1$.

4.2. The line-homogeneous Universe:

$$R\ (t)\ w_{,gg}/w^2 = f_L(g), \tag{4.2}$$

where

$$g = x/y, \quad w\ (t,g) = V/y, \tag{4.3}$$

and $f_L(g)$ is an arbitrary function. These solutions generalize also only the flat FLRW model. Note that (4.1) and (4.2) is in fact the same equation, only the geometrical meaning of the independent variable and of the unknown function is in each case different. The equation will appear once more below, in a still different disguise.

5. THE SOLUTIONS FOR THE GENERIC CASE, $F_1 \cdot F_2 \cdot F_3 \neq 0$

In this case, the equations resulting from (3.6) are solved by:

$$G_1 = F_1(F_3/F_2 - F_2/F_3) \ ,$$

$$G_3 = F_3(F_2/F_1 - F_1/F_2) \ , \tag{5.1}$$

while those resulting from (3.7), with the substitutions:

$$F_3 = P_3F_1 \ , \quad F_2 = P_2F_1 \tag{5.2}$$

reduce to the following set:

$$-P_{2,x} - P_{2,y}/P_2 + P_3(P_2 - 1/P_2)_{,z} = 0,$$

$$-P_{3,x} - P_{3,z}/P_3 + P_2(P_3 - 1/P_3)_{,y} = 0, \tag{5.3}$$

$$-P_{3,y} + P_3P_{2,y}/P_2 + P_{2,z} - P_2P_{3,z}/P_3 + (P_3/P_2 - P_2/P_3)_{,x} = 0 .$$

Note that these are 3 equations for only 2 functions, since F_1 does not appear here. The solutions of (5.3) must then be substituted into the integrability conditions (3.5) (of which only two will remain in virtue of (5.3)), and after (3.5) are solved, the solution of (3.3) - (3.4) is guaranteed to exist (but must be found in the next step!). So far, only one special solution of (5.3) was found which results when $P_{2,z} = P_{3,y} = 0$ and the first two equations decouple. With that, (3.3) - (3.4) reduce to the following single ordinary differential equation:

$$R(t)\, V,_{uu} / V^2 = f_S(u) \;, \tag{5.4}$$

where

$$u = x^2 + y^2 + z^2 \;, \tag{5.5}$$

and $f_S(u)$ is an arbitrary function. We call the resulting solution a spherically symmetric Universe. Its subcases include the spherically symmetric Stephani solution (which results with $f_S = 0$, $V,_u \neq 0$) and <u>all</u> the FLRW solutions (which result if in addition $V,_t = 0$). Note that (5.4) is again the same equation as (4.1) and (4.2), with a still different meaning of the variable.

More solutions of (5.3), and thus of (3.3) - (3.4), very possibly exist, but are not explored so far; the work is in progress.

6. AN EXAMPLE OF A SOLUTION OF (4.1)

Let $f_P = C_P$ = const in (4.1). Then:

$$V,_z{}^2 = (2/3)\; C_P V^3/R(t) + K(t) \;, \tag{6.1}$$

where $K(t)$ is an arbitrary function. Equation (6.1) reminds of the defining equation of the Weierstrass elliptic function $\wp(z)$[13]:

$$\wp,_z{}^2 = 4\wp^3 - g_2\wp - g_3 \;, \tag{6.2}$$

where g_2, g_3 = const. In (6.1) the second term is missing, and that is a pity because with $g_2 \neq 0$ some solutions of (6.2) are periodic in z. Let us then change the choice of $f_P(z)$. Let

$$f_P(z) = [6 - (1/2)\; g_2 / J^2(z)]\; R(t_o) \;, \tag{6.3}$$

where g_2 and t_o are arbitrary constants, and $J(z)$ is defined by the equation

$$J,_{zz}/J^2 = 6 - (1/2)\; g_2/J^2. \tag{6.4}$$

Eq. (6.4) results from the Weierstrass equation (6.2) on differentiating with respect to z, so $J(z) = \wp(z)$ is one of the solutions of (6.4) and can thus be periodic in z. Then, with (6.3), V

$= J(z)$ is a solution of (4.1) at $t = t_o$, and so $V(t_o,z)$ is periodic in z. With such $V(t,z)$ the mattter-density

$$8\pi G\rho/c^2 = -2f_p V^3/R^3 - 3V_{,z}{}^2/R^2 + 3/F^2 \qquad (6.5)$$

(where F is the function from (3.2)) is periodic in z at $t = t_o$.

7. COMMENTS

The previous section shows that initial conditions can be set up in such a way that the matter-density in the model will be periodic in one spatial variable at $t = t_o$. What happens with this structure as time proceeds, remains to be investigated. The structure is not complicated enough to be a model of the real Universe (but is more general than the flat FLRW model!), nevertheless, can be useful as an explicit example for studying the consequences of averaging physical quantities over space.

Analogous solutions exist for (4.2) and (5.4). In (4.2), $w(t_o,g)$ can be periodic in $g = x/y$. In (5.4), $V(t_o,u)$ can be periodic in $u = x^2 + y^2 + z^2$. In both cases, however, $\rho(t_o,g)$ and $\rho(t_o,u)$ are not periodic, but only periodically modulated (with a variable amplitude). These two solutions thus represent Universes which are totally inhomogeneous at $t = t_o$.

It would be desirable to obtain a further generalization of the solution from sec. 6 which would be periodic in all three spatial coordinates. This remains a challenge for the future.

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