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CONTENTS

GENERALIZED COSMOLOGICAL MODELS

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ABSTRACT

Two classes of generalizations of the Friedman-Lemaitre models within Einstein's theory of relativity are presented. One is obtained by assuming that the spacetime is a congruence of homogeneous and isotropic hypersurfaces whose orthogonal trajectories coincide with matter world-lines - without assuming that the spacetime inherits these symmetries. In the resulting Stephani Universe the topology of the spatial sections may vary with time. A second class is obtained by assuming that the spatial sections are conformal to flat spaces. The field equations are in this case reduced to three linear partial differential equations of first order for two functions. Only the spherically symmetric subclass of a possibly rich family of solutions is investigated. In one solution the density of matter contains terms which are spatially periodic at a certain instant.

1. THE MOTIVATION FOR THIS INVESTIGATION

The classical Friedman-Lemaitre-Robertson-Walker cosmological models proved successful in explaining and predicting several physical properties of our Universe. There is, however, one problem that cannot be exactly solved in them, and that is the formation of galaxies. All attempts to describe that process[1] were based on perturbations of the FLRW models. This fact alone shows that more general solutions of the Einstein´s equations would be useful. More arguments for such generalizations may be found in the work of Ellis[2].

Since, however, the FLRW models were so successful, it is reasonable to expect that the more general and more realistic models will contain them as special cases, i.e. first approximations within the collection of solutions of the Einstein´s equations. This note will show how two families of such generalized models may be obtained by gradually weakening the assumptions underlying the FLRW solutions. In one of them, called the Stephani Universe[3-5], the spatial sections are assumed to be still homogeneous, isotropic and orthogonal to the world-lines of matter, but the whole spacetime is not required to inherit the symmetry of the sections. The resulting solution has in general no symmetry at all, and the topology of the sections may be changing with time.

In another family, the sections are assumed to be conformal to a flat space. The second family thus contains the first one. The field equations could not be so far solved in the most general case, and only two special spherically symmetric solutions were found. In one of them, the matter density contains terms which are periodic or periodically modulated in the <u>radial</u> coordinate. This solution (which contains the FLRW solutions as special cases) thus represents a Universe with an evolving structure. Being spherically symmetric and inhomogeneous, it violates the Copernician principle, but may represent a first step in constructing a model endowed wih <u>discrete</u> homogeneity rather than a continuous one[2].

2. CASE I: INTRINSICALLY HOMOGENEOUS AND ISOTROPIC SPACE-TIMES.

In the FLRW models many important conclusions for the whole Universe follow from the distribution of matter in a single 3-space t = const. Among other things, the matter density in a 3-space should decide whether our Universe is closed, spatially flat or open. This raises the question: to what extent is the 4-dimensional geometry of our space-time determined by a given class of 3-dimensional geometries of the t = const spaces? Since it is commonly believed that these 3-spaces are homogeneous and isotropic, we shall at first keep that assumption and ask the question: what kind of spacetimes can have such t = const sections?

Spacetimes which contain subspaces of a definite symmetry group were called by Collins[6] "intrinsically symmetric". We shall thus deal with "intrinsically homogeneous and isotropic" spacetimes. It should be remembered that there exist the 9 Bianchi types of homogeneous spaces, 3 of which can simultaneously be isotropic. Consequently, we did not specify the 3-geometries uniquely by our assumption, and it should not be surprising when the consequences of this nonuniqueness show up.

3. THE SPACETIME METRIC IN CASE I.

In order to make the calculations easier, we shall make two more assumptions in addition to the assumption made in sec. 2 to which we refer by (1):

(2) The lines of the t-coordinate are orthogonal to the spaces t = const.

(3) Matter flow-lines coincide with the t-lines.

We also assume that:

(4) The Einstein´s field equations hold, the source being a perfect fluid.

However, we shall not assume anything about the symmetry of the whole spacetime.

The assumption (1) implies that the metric form of the 3-spaces t = const can be represented as:

$$ds_3^2 = (R^2 / V^2)(dx^2 + dy^2 + dz^2), \tag{3.1}$$

where

$$V = 1 + \frac{1}{4} k \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \right], \tag{3.2}$$

R, k, x_0, y_0, z_0 being arbitrary constants. Since this is supposed to be a section t = const of a spacetime, each constant must be understood as a momentary value of a function of t at $t = t_0$. Consequently, the 4-dimensional metric will contain functions $R(t)$, $k(t)$, $x_0(t)$, $y_0(t)$ and $z_0(t)$ in place of the constants R, k, x_0, y_0, z_0.

Assumption (2) now implies $g_{tx} = g_{ty} = g_{tz} = 0$, so our metric will be:

$$ds^2 = D^2 dt^2 - (R^2 / V^2)\,(dx^2 + dy^2 + dz^2), \tag{3.3}$$

where $D = D\,(t, x, y, z)$, and V is given by (3.2) with the constants replaced by the appropriate functions.

When this metric is substituted into the Einstein´s field equations, it produces the following solution:

$$D = F(t)\,\frac{R}{V}\,\frac{\partial}{\partial t}\left(\frac{V}{R}\right), \tag{3.4}$$

$$k\,(t) = (C^2 - 1/F^2)\,R^2\,, \tag{3.5}$$

where $F(t)$ and $C(t)$ are new arbitrary functions of t. $R(t)$, $x_0\,(t)$, $y_0\,(t)$ and $z_0\,(t)$ are not determined by the field equations. The matter-density and the pressure are given by

$$\begin{aligned} \kappa\varepsilon &= 3C^2\,(t), \\ \kappa p &= -\,3C^2 + 2C\dot{C}\,\frac{V}{R}\Big/\frac{\partial}{\partial t}\left(\frac{V}{R}\right)\,, \end{aligned} \tag{3.6}$$

$\kappa = 8\pi G/c^4$. The function F is related to the expansion scalar of the fluid flow by $\theta = -3/F$.

This solution was found by Stephani in 1967[3], but not discussed by then in the cosmological context.

4. PROPERTIES OF THE STEPHANI SOLUTION

The Stephani solution shares the following properties with the FLRW solutions:

1. The matter source is a perfect fluid moving with zero shear and zero rotation.

2. The metric is conformally flat (moreover, it is the most general conformally flat solution with an expanding perfect fluid source [7]).

The solution is different from FLRW in the following respects:

1. In general it has no symmetry at all.

2. The matter moves with acceleration, i.e. not on geodesic lines.

3. No equation of state of the simple form $p = p(\varepsilon)$ holds, pressure depends also on spatial position (it simply means that temperature must enter the equation of state and that the temperature varies with spatial position).

The characteristic property of this Stephani solution is the dependence of k on time in such a way that the sign of k is not determined. Since the sign of k is the sign of spatial curvature of the sections t = const, it means that in this Universe some spatial sections may have positive curvature and so be closed, while some others will have negative curvature and be open. An example of a spacetime with such a topology of spatial sections will be given below. Let us only note that the FLRW models are contained in this one as special cases. Namely, the Stephani solution reduces to a FLRW solution under any of the following conditions:

1. The functions k, x_0, y_0, z_0 are all constant (the time-derivatives of these functions are responsible for the

lack of symmetry; with only x_0, y_0, z_0 being constant and $k_t \neq 0$ the solution becomes spherically symmetric but inhomogeneous).

2. The acceleration field vanishes (i.e. matter moves on geodesics).

3. The equation of state is of the form $p = p(\varepsilon)$, i.e. does not depend on position.

Stephani has shown that this solution may be embedded in a flat 5-dimensional space. The embedding is a convenient method of studying global properties of a spacetime. However, it would be a difficult study for the Stephani solution in its full generality because of the 6 arbitrary functions. It is more instructive to consider the simpler special case C = const. The Stephani solution reduces then to the deSitter solution. If we further simplify the foliation by assuming $x_0 = y_0 = z_0 = 0$, R = const, k = -t, then the embedding can be described by simple formulae and shown on a figure. The metric of the 5-space is then:

$$dS^2 = dZ^2 - dX^2 - dU^2 - dW^2 - dY^2 , \qquad (4.1)$$

while the equation of the deSitter (sub-)space is:

$$Z^2 - X^2 - U^2 - W^2 - (Y - 1/C)^2 = -1/C^2 , \qquad (4.2)$$

or, in parametric form:

$$Z = R(x^2 + y^2 + z^2)(C^2R^2 + t)^{1/2}/2V, \qquad (4.3)$$

$$(X, U, W) = (R/V)\,(x, y, z), \qquad (4.3)$$

$$Y = CR^2\,(x^2 + y^2 + z^2)/2V\,. \qquad (4.3)$$

From these equations it can be seen that the whole space-time is a 4-dimensional one-sheet hyperboloid (eq. 4.2) while its sections t = const are intersections of the hyperboloid with the planes $Z/Y = (C^2R^2 + t)^{1/2}/CR$ (= const at t = const). It can be seen that for $t = -C^2R^2$ the intersection is a 3-sphere, for $-C^2R^2 < t < 0$ the intersections are 3-dimensional ellipsoids, for $t = 0$ the intersection is a 3-dimensional paraboloid, and for $t > 0$ the intersections are 3-dimensional two-sheet hyperboloids. The projection of the deSitter hyperboloid on the plane X, U, W = const is shown in the figure.

The proper times between different t = const spaces along a t-line and the proper distances within each single t = const space may be calculated or estimated in the general case. In this way, it can be verified that the figure faithfully represents the topology of the general Stephani solution, except for one aspect. The singularity seen at $Z = Y = 0$ is in general a true curvature singularity that occurs at different spatial positions for every t. It is an additional singularity to the one predicted by the Hawking-Penrose theorems, and it can be avoided if the functions k, R and their time-derivatives obey certain inequalities[5]. The inequalities can be fulfilled if either k is always positive or p is negative in some range of (x, y, z) at every t for which $k(t) < 0$.

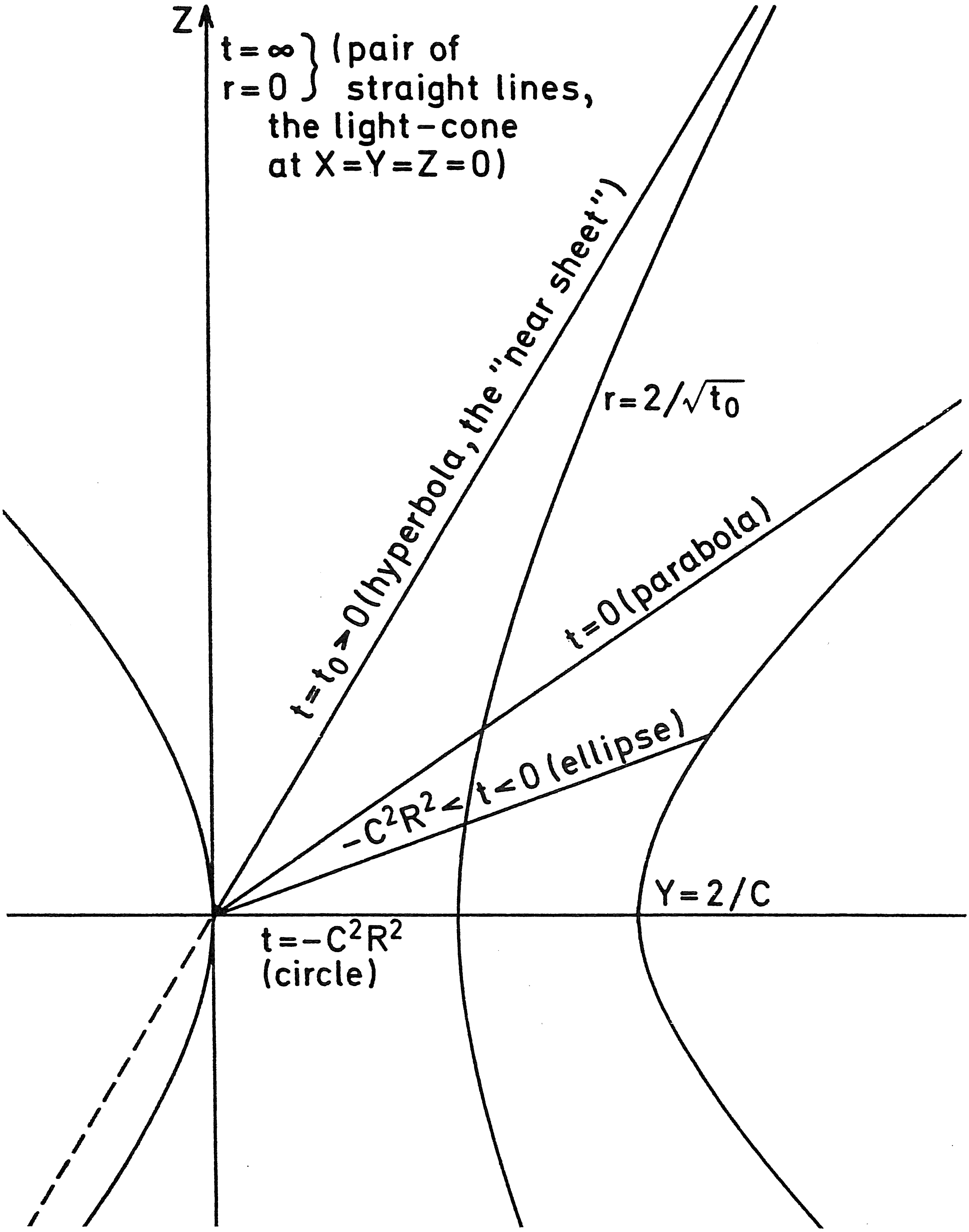

Fig. 1. Projection of the deSitter manifold onto the (Y, Z) plane. The parametrization (4.3) covers only the two sectors where YZ > 0. (Adapted from Ref. 5 with the permission of the Plenum Publishing Corporation).

5. CASE II: SPACETIMES WITH CONFORMALLY FLAT SECTIONS

The Stephani solution is still not general enough in order to cure the main failure of the FLRW models: the density of matter in it is spatially homogeneous, so no structures can develop. Further (or other) generalizations are necessary.

It would be nice to proceed from a set of physically motivated assumptions. However, none would be obvious. The guiding principle which I want to retain is that the generalized models searched for should contain the FLRW ones as special cases and should be mathematically as close to them as possible, in order to avoid too great complexities for the beginning. A reasonably simple assumption is: let all the t = const spaces be conformally flat. While being a rather formal one, this assumption is fulfilled by the FLRW models, and adds only one more unknown function to the metric considered in sec. 3 (we shall retain the other 3 assumptions made at the beginning of sec. 3). Thus the metric form we shall start with is:

$$ds^2 = D^2 dt^2 - V^{-2} R^2(t) (dx^2 + dy^2 + dz^2), \qquad (5.1)$$

where now both D and V are unknown functions of (t, x, y, z). The velocity field of the fluid will, as before, have the form:

$$u^\alpha = D^{-1} \delta^\alpha_0 . \qquad (5.2)$$

With such u^α, the field equations $G_{0i} = 0$ yield at once:

$$D = F(t)\,\frac{R}{V}\,\frac{\partial}{\partial t}\left(\frac{V}{R}\right), \tag{5.3}$$

just as in the Stephani model. From this equation alone it follows that the shear of the fluid flow vanishes while the scalar of expansion equals again $\theta = -3/F = \theta(t)$, i.e. is homogeneous and isotropic. In order to get rid of this residual regularity we should thus give up the assumption which produced (5.3), and that was (5.2). We can have shear or inhomogeneous expansion only if we consider flows in which the velocity field is not tangent to the t-lines of the metric (5.1).

The other field equations imply then:

$$R\,V,_{ij} / V^2 = F_k(x, y, z)\ , \tag{5.4}$$

$$R\,(V,_{ii} - V,_{jj}) / V^2 = G_k(x, y, z) \tag{5.5}$$

(no summation), where F_k and G_k are arbitrary functions of x, y, z (independent of t), and the indices i, j, k run cyclically through the values 1, 2, 3. The Stephani solution follows uniquely when $F_k = G_k = 0$.

The equations (5.3), (5.4) and (5.5) guarantee that $G_{\alpha\beta} = \kappa(\varepsilon + p)u_\alpha u_\beta - pg_{\alpha\beta}$. The remaining field equations simply define pressure and energy-density. However, the functions F_k and G_k must obey several integrability conditions in order that (5.4) and (5.5) are solvable.

6. THE RELEVANT SET OF SUBCASES

Since $G_2 \equiv G_1 + G_3$, there are only five equations in the set (5.4) - (5.5). Their integrability conditions have the form:

$$A^i_L V,_i/V = B_L, \qquad (6.1)$$

where A^i_L is a 5×3 matrix and B_L is a 5-vector, both composed of the functions F_k, G_k and their derivatives, thus being independent of t. At most 3 of these 5 equations can be independent, otherwise the set is not even algebraically solvable for $V,_i$. However, even this would be too much. It can be shown by elementary considerations that if a subset of 3 independent equations is contained in (6.1), then either the FLRW solutions result or else $V = R(t)\cdot H(x,y,z)$. In the second case it can be further shown that the matter density does not depend on time and the expansion scalar is zero. Such a solution is not interesting from the point of view of cosmology, and we shall not consider it here.

Thus we can expect a generalization of the Stephani solution only when no subset of 3 equations can be chosen from (6.1) which would be solvable algebraically for $V,_i$. This means that every 3×3 matrix contained in A^i_L should have a vanishing determinant. Setting aside the special cases when $F_1 F_2 F_3 = 0$ this condition is equivalent to:

$$G_1 = F_1 (F_3 / F_2 - F_2 / F_3) ,$$
$$G_3 = F_3 (F_2 / F_1 - F_1 / F_2) . \tag{6.2}$$

The special cases $F_1 F_2 F_3 = 0$ are easily shown to lead to solutions which do not contain the FLRW ones and so are not interesting from our point of view.

With (6.2) fulfilled, the equations (6.1) do not determine all the derivatives $V_{,i}$. That means, at most two of them can be independent if the set is not to be contradictory. This imposes 3 more equations on the functions F_k.

Let us denote:

$$F_2 = P_2 F_1 ,$$
$$F_3 = P_3 F_1 . \tag{6.3}$$

The 3 equations are then:

$$- P_{2,x} - P_{2,y} / P_2 + P_3 (P_2 - 1/P_2)_{,z} = 0, \tag{6.4}$$

$$- P_{3,x} - P_{3,z} / P_3 + P_2 (P_3 - 1/P_3)_{,y} = 0, \tag{6.5}$$

$$-P_{3,y} + P_3 P_{2,y}/P_2 + P_{2,z} - P_2 P_{3,z}/P_3$$
$$+ (P_3/P_2 - P_2/P_3)_{,x} = 0 . \tag{6.6}$$

These equations connect P_2 and P_3, the function F_1 remains arbitrary. There are again several special cases to be considered separately, and not all of them have been investigated as yet. One case, however, is seen at once to be particularly simple: if $P_{2,z} = P_{3,y} = 0$, then (6.4) and (6.5) become independent and can be integrated. This case leads to two interesting solutions.

7. ANOTHER GENERALIZATION OF THE FLRW MODELS

The solutions of (6.4) and (6.5) must also obey (6.6), this is why the result of the ansatz $P_{2,z} = P_{3,y} = 0$ is so special:

$$P_2 = x/y ,$$
$$P_3 = x/z . \tag{7.1}$$

With such P_2 and P_3 the whole set (6.1) reduces to just two equations which are easily solved, the solution is:

$$F_1 V^2 = yz\, U(t,u)\ ,$$
$$u \stackrel{\text{def}}{=} x^2 + y^2 + z^2\ , \qquad (7.2)$$

where U is an arbitrary function of its two arguments. We recall that (7.2) solves only the integrability conditions of the equations (5.4) - (5.5) which are still to be solved. With (7.1) and (7.2) it can be shown from (5.4) - (5.5) that $F_1 = yz\, f(u)$ where $f(u)$ is an arbitrary function, and so $V = V(t,u)$, i.e. the spacetime is spherically symmetric, see (5.1) - (5.3). The whole set (5.4)- (5.5) reduces then to the single equation:

$$R(t)\, V,_{uu} / V^2 = f(u)\ . \qquad (7.3)$$

A general solution of this equation is so far unknown[7]. However, two special cases can be discussed in more detail. Let first:

$$f(u) = B = \text{const.} \qquad (7.4)$$

Then (7.3) has a first integral:

$$V_u^2 = 2BV^3/3R(t) + S(t)\ , \qquad (7.5)$$

where S(t) is an arbitrary function. The solution of this equation may be represented by:

$$\int [2BV^3/3R(t) + S(t)]^{-1/2} \, dV = u + 4/k(t), \qquad (7.6)$$

where k(t) is another function of time. From this form, all the FLRW solutions are seen to be contained in this one as the special case $B = 0$, $k = \mathrm{const}$, $S = k^2/16$. The solution (7.6) is spherically symmetric, so it cannot contain the most general Stephani Universe, but with $B = 0$, $S = k^2/16$ and $k \neq \mathrm{const}$ it reduces to the spherically symmetric subcase $x_0 = y_0 = z_0 = 0$ of (3.1) - (3.5). The matter density corresponding to (7.5) is:

$$\kappa\varepsilon = 12[VV,_u - S(t)u] / R^2(t) + 3/F^2(t), \qquad (7.7)$$

and so is seen to be spatially inhomogeneous - and changing with time. This solution thus describes an "exact perturbation" of the FLRW models in which there exist evolving structures.

One can also obtain a more interesting solution with evolving structure. Note that eq. (7.5) is similar to the equation:

$$\wp_u^2 = 4\wp^3 - g_2\wp - g_3 \qquad (7.8)$$

which defines the Weierstrass elliptic function[9]. This function is periodic in its argument u, provided $g_2 \neq 0$ (*), while (7.5) corresponds to $g_2 = 0$. We can, however, choose f(u) in another way. Let:

$$f(u) = [6 - g_2 / 2 J^2(u)]\, R(t_0) \,, \tag{7.9}$$

where g_2 = const, $t = t_0$ is an arbitrarily chosen instant of time, and J(u) is defined by the differential equation:

$$J_{,uu}^2 / J = 6 - g_2 / 2 J^2(u) \,. \tag{7.10}$$

It is easily seen that any solution of (7.8) fulfills also (7.10), and so (7.10) contains periodic functions among its solutions. Now we have, in virtue of (7.9) and (7.3):

$$V_{,uu}^2 / V = [R(t_0)/R(t)]\, [6 - g_2 / 2 J^2(u)] \,. \tag{7.11}$$

(*) It can be easily seen that V given by (7.5) or $\wp$ given by (7.8) with $g_2 = 0$ are not periodic. If we replace u by t and $\wp$ by r, then (7.8) becomes formally identical to the energy integral of the Newtonian equation of radial motion in the potential $U(r) = -C r^3$. This motion is evidently not periodic. I am grateful to Pascal Nardone for this neat argument.

Thus $V(t_0,u) = J(u)$ is a solution of (7.11) at $t = t_0$, i.e. $V(t_0,u)$ can be spatially periodic. But even then, the matter-density:

$$\kappa\varepsilon = (4/R^2)\,[3VV,_u^2 + 2uf(u)V^3/R - 3uV,_u^2] + 3/F^2(t)\ ,\qquad(7.12)$$

is not strictly periodic at $t = t_0$. The second and third term in brackets are periodically modulated, but their amplitudes grow with u. Whether there exist solutions with spatially periodic matter distribution in this class remains to be seen.

8. POSSIBLE FURTHER GENERALIZATIONS

The problem of existence of solutions with spatially periodic matter density is interesting, since spatial periodicity is a way of reconciling the evident inhomogeneity of the Universe in a small scale[10)] with the postulated large-scale homogeneity - by means o a <u>discrete</u> symmetry group acting on spatial sections of the spacetime[2)]. Then, however, V (t, x, y, z) would have to be a periodic function in each of the three coordinates. The existence of the solution (7.11) may be an indication that the search for such periodic solutions is not totally futile. Some progress may be expected even within the class of spacetimes given by (5.1). We were directed towards spherically symmetric solutions by the assumptions $P_2,_z = P_3,_y = 0$. Without these assumptions, we are still left with a possibly large family of less symmetric solutions.

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