A GENERALIZATION OF THE LEMAITRE MODELS

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Abstract A generalization of the Friedman-Lemaitre -Robertson-Walker (FLRW) models is obtained by weakening the assumptions under which they are derived from Einstein´s relativity. It is assumed that each section t = const is homogeneous and isotropic while the space-time itself not necessarily has any symmetry. The resulting Stephani Universe has an undetermined function of time in place of the constant curvature index k. In this Universe, some spatial sections may be open while others will be closed. Its geometrical picture is presented and its physical properties are discussed.

1. GENERALIZATIONS ARE COMPATIBLE WITH OBSERVATIONS.

The Friedman-Lemaitre-Robertson-Walker (FLRW) solutions of Einstein´s field equations (1 - 4) were derived under the very strong assumptions that the space-time is homogeneous and isotropic. These assumptions were not meant to reflect our knowledge about the Universe, but rather our ignorance: at that time (1930-ies) no structures larger than galaxies were known. The homogeneous and isotropic distribution of galaxies was thus a reasonable first hypothesis which at the same time made Einstein´s equations tractable.

The models proved successful in describing several observable properties of the Universe, like Hubble´s expansion law, the abundance of helium or the microwave background radiation. These successes are often understood as confirmations of the underlying assumptions.

A. Berger (ed.), The Big Bang and Georges Lemaitre, 63–73.

In fact, they only confirm that the Universe was much hotter and denser in the past than it is now and that it was very nearly isotropic at the time when the radiation last interacted with massive particles. Even within these classical models massive particles and radiation are considered as two independent components of matter which decoupled in the moment of last scattering and later evolved independently. Whatever happened to particles afterwards, did not affect the distribution of radiation. Therefore, the isotropy of radiation does not force upon us a model in which all matter is distributed so symmetrically.

Once isotropy is given up, the requirement of homogeneity is not compelling anymore. The Universe is assumed homogeneous because it would be unnatural if it were spherically symmetric only around us (5), but if it is not spherically symmetric at all, it might be inhomogeneous as well. This statement does not speak against the copernician philosophy. According to it, no place in the Universe should be preferred. This does not mean that all the places in the Universe should be exactly identical. The latter assumption fulfills the former, but is much stronger (see also Ref. 6).

A purely theoretical argument also shows that more general models of the Universe can be reconciled with the existing data (cf Fig. 1, in Ref. 7). Only the events lying on our past light cone are directly observed, and only directions to them can be measured with a satisfactory precision. All other data needed to calculate the spatial distribution of matter are inferred therefrom through a model-dependent procedure: 1. Through each event on the light cone we draw a world line representing the history of that portion of matter, e.g. a galaxy (the equations of those lines can only be calculated given a specific class of spacetimes); 2. Through the vertex of the light cone we draw a hypersurface S of events simultaneous with "now" (Even within a fixed model this depends on the reference system chosen. The reference system is usually attached to a physical structure in the spacetime, e.g. the congruence of matter world-lines); 3. The points of intersection of the world-lines with the hypersurface S represent the positions of the galaxies now (These positions depend on the slopes of the matter world-lines, i.e. on the velocity of expansion, given the model and given S. This velocity can be calculated from the observed redshift - provided we know precisely what part of the redshift is of cosmological origin). 4. Only now can we calculate the spatial distribution of matter. Thus a model is assumed before any observations are taken into account. It can be confirmed or refuted by these observations, but is in no way implied by them (see also Refs 6, 8, 9).

2. GENERALIZATIONS ARE IN FACT NECESSARY.

According to present data, galaxies are grouped into clusters and shells surrounding voids which contain no visible matter at all (10). Thus the Universe might possibly be homogeneous only on still larger scales (if at all). Such a large scale homogeneity coupled with small scale inhomogeneity is not properly described by a spacetime with a continuous transitive group of symmetry (curve a in Fig. 1) of which the FLRW spacetimes are examples. A more appropriate description would be a spacetime with a discrete group of symmetry in which matter density would be given by a function like curve b in Fig. 1 (see similar remarks by Ellis (6)). Such distribution of matter does not distinguish any single observer because, if the space is infinite, there exist infinitely many identical copies of any chosen finite portion of matter distributed regularly. Such a solution can only be found if the assumption of continuous homogeneity is relaxed altogether.

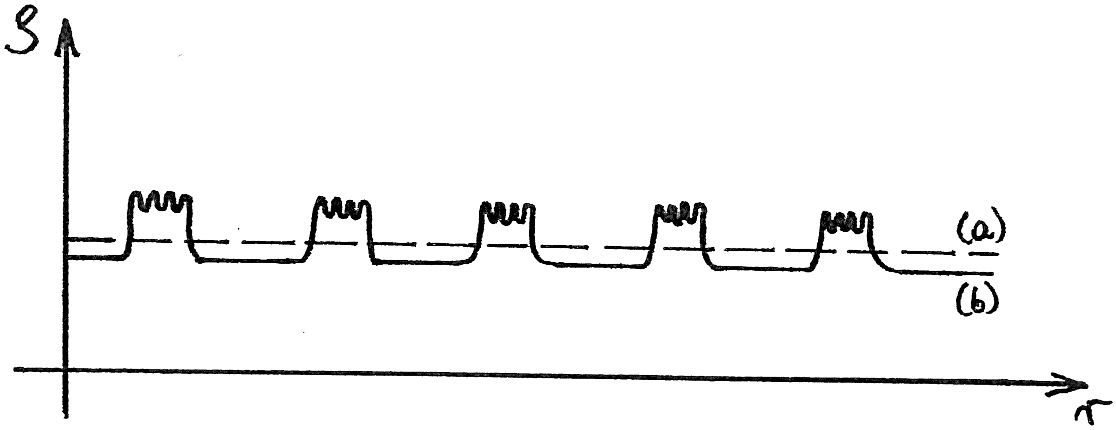

Fig 1. Matter density vs position in a 3-space that is homogeneous with respect to (a) a continuous group of symmetry, (b) a discrete group of symmetry.

Moreover, the FLRW models taken literally tell us that no galaxies may ever have formed out of a homogeneous and isotropic background. All theories of galaxy formation must consider perturbations of the FLRW models (see e.g. (11)). If we have to go beyond the FLRW models, it is equally reasonable to consider exact generalizations instead of approximate perturbations.

This author is in a definite minority, but not alone with his criticism of "standard cosmology". Similar concerns were expressed by Ellis (6, 8, 9, 12-14), MacCallum (15) and Mashhoon (16, see also this volume).

Since the FLRW models proved so successful, the more general new models should contain them as special cases, i.e. as first approximations. This paper will show how a certain generalization results if the assumptions underlying the FLRW models are slightly relaxed. This generalization does not go sufficiently far in order to be free from the above mentioned weaknesses. Its existence proves however that more general solutions can still be reasonably simple.

3. ASSUMPTIONS.

The considerations of the previous sections show that what is checked against astronomical observations is the 3-dimensional space t = now rather than the whole spacetime. It is then a natural question, to what extent the 3-geometries of the spaces t = const determine the 4-geometry of our spacetime. Let us assume, as is commonly done, that:

1. Each 3-space t = const is homogeneous and isotropic,
2. The spaces are orthogonal to the family of t-coordinate lines,
3. Matter moves along the t-lines,
4. The Einstein´s field equations are fulfilled, the source being a perfect fluid,

but let us consider the possibility that:

5. The spacetime not necessarily has any symmetry.

4. THE SOLUTION.

The assumptions 1 to 5 produce the following solution of the Einstein´s equations (17):

$$ds^2 = D^2 dt^2 - (R^2 / V^2)(dx^2 + dy^2 + dz^2), \qquad (4.1)$$

$$D = F \frac{R}{V} \frac{\partial}{\partial t} \left(\frac{V}{R} \right), \qquad (4.2)$$

$$V = 1 + \frac{1}{4} k \{ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \} \quad (4.3)$$

$$k = (C^2 - 1/F^2) R^2, \quad (4.4)$$

$$\kappa \varepsilon = 3C^2, \quad (4.5)$$

$$\kappa p = - 3C^2 + 2C \frac{dC}{dt} \frac{V}{R} \Big/ \frac{\partial}{\partial t} \left(\frac{V}{R} \right), \quad (4.6)$$

$$a^0 = 0, \qquad a^i = (V^2/DR^2) D_i, \qquad i = 1, 2, 3. \quad (4.7)$$

where F, R, C, x_0, y_0, z_0 are arbitrary functions of time, ε is the energy density, p is the pressure, and a is the acceleration field of the fluid flow.

This solution was first found by Stephani (18) in 1967, but was not investigated from the point of view of cosmology.

5. LOCAL PROPERTIES OF THE SOLUTION.

The solution has in general no symmetry at all. Its most striking property is the fact that k is a function of t, the sign of k being not determined. Since k is the curvature index of the 3-spaces t = const, one sees that in this spacetime some spacelike sections have positive curvature (and so should be closed) while some others have negative or zero curvature (and so should be open). Other differences with the FLRW solutions are the following:

1. Matter moves with acceleration, i.e. not on geodesic lines.

2. The equation of state is not of the form $\varepsilon = \varepsilon(p)$, but depends on the position in the space: $p = p(\varepsilon, x, y, z)$.

This last property means that a single thermodynamic function of state (e.g. pressure) does not suffice to describe matter in this model, at least one other function is necessary, e.g. temperature which would have different values in different places.

The Stephani Universe reduces to a FLRW model when any one of the following situations occurs:

1. The functions x_0, y_0, z_0 and k are constant.

2. The acceleration field vanishes (i.e. matter moves on geodesics).

3. The equation of state is of the form $\varepsilon = \varepsilon(p)$, i.e. it does not depend on position.

This solution is conformally flat, and moreover it is the most general conformally flat solution with a perfect fluid source and nonvanishing expansion (19).

6. GLOBAL PROPERTIES OF THE STEPHANI UNIVERSE.

Stephani has shown (18) that this solution can be embedded in a flat five-dimensional space. To construct the embedding explicitly would in general be too difficult because of the 6 arbitrary functions of time. It was more instructive to study a special case in which the embedding could be performed explicitly.

Such a special case results when C = const; the Stephani Universe reduces then to the deSitter solution. It was further assumed $x_0 = y_0 = z_0 = 0$, R = const, k = -t. In the case C = const these additional assumptions amount just to a choice of a simpler coordinate system (foliation).

The deSitter manifold is then a 4-dimensional one-sheet hyperboloid embedded in a 5-dimensional pseudoeuclidean space. The metric form of the 5-space is:

$$dS^2 = dZ^2 - dX^2 - dU^2 - dW^2 - dY^2, \tag{6.1}$$

while the equation of the deSitter hyperboloid is:

$$Z^2 - X^2 - U^2 - W^2 - (Y - 1/C)^2 = -1/C^2, \tag{6.2}$$

or, in parametric form:

$$Z = R(x^2 + y^2 + z^2)(C^2R^2 + t)^{1/2} / 2V \tag{6.3}$$

$$(X, U, W) = (R/V)(x, y, z) \tag{6.4}$$

$$Y = CR^2(x^2 + y^2 + z^2) / 2V \tag{6.5}$$

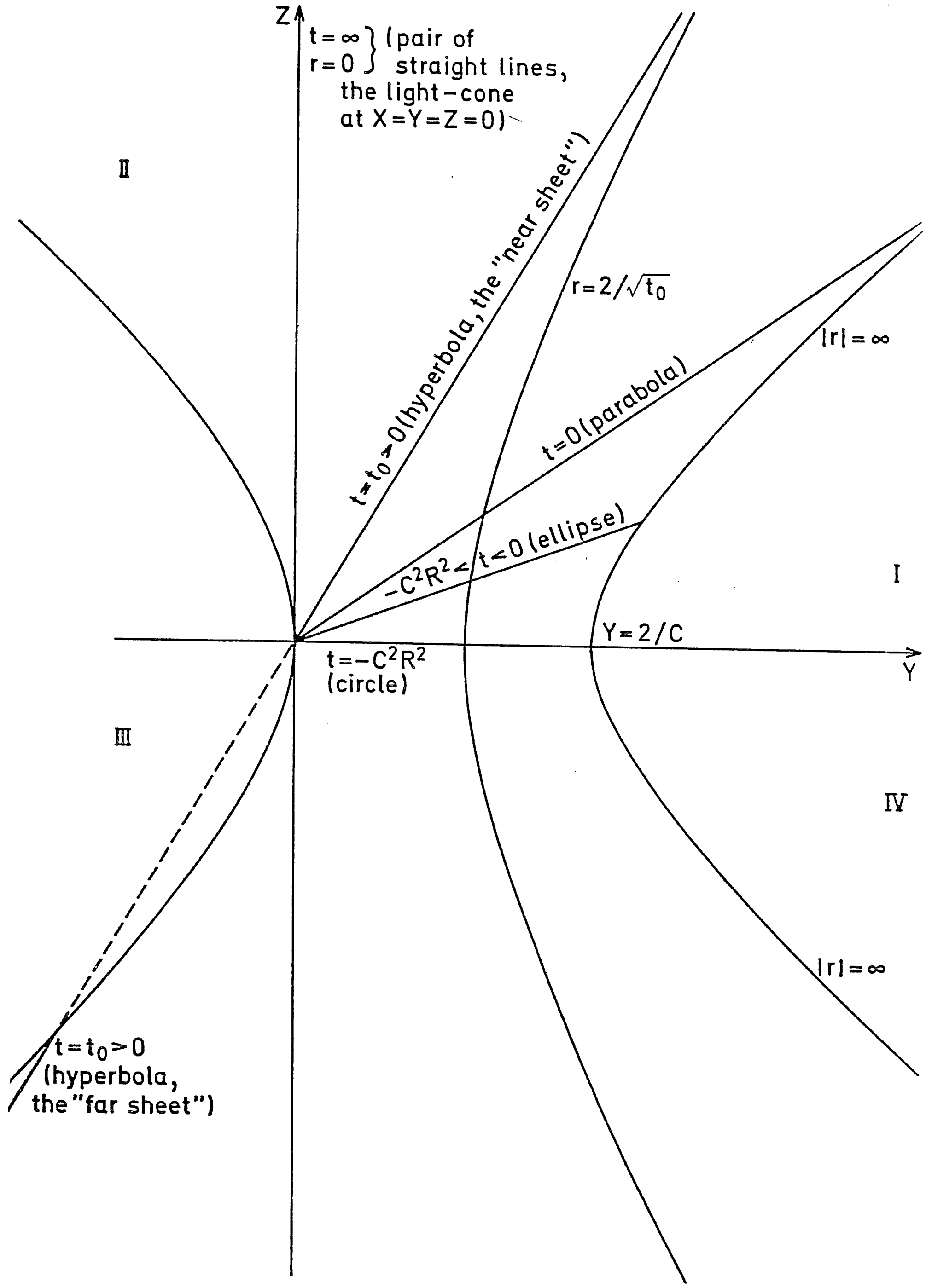

Fig. 2. Projection of the deSitter manifold onto the (Y, Z) plane. Only sectors I and IV are covered by the parametrization (6.3) - (6.5). See text for more remarks. (Adapted from Ref. 20 with the permission of the Plenum Publishing Corporation).

The projection of the hyperboloid (6.2) onto the (Y, Z) plane in the space (6.1) is shown in Fig. 2. On the figure one can see that a spacetime of a simple topology can result from the foliation introduced in sec. 3. The sections t = const of the spacetime are intersections of the hyperboloid (6.3) - (6.5) with the hyperplanes Z/Y = const. They all contain the (X, U, W) space (the X axis in Fig. 2) and their tilt to the (X, U, W, Y) hyperplane (the (X, Y) plane in Fig. 2) is determined by $k(t) = -t$. With $-C^2R^2 \le t < 0$ we have $k(t) > 0$ and the tilt of the t = const hyperplanes is such that their intersections with the hyperboloid (6.2) are 3-ellipsoids (ellipses in Fig. 2) - closed spaces of positive curvature (in the special case $t = -C^2R^2$ it is a 3-sphere). With $t = 0$ we have $k = 0$ and the intersection is a 3-paraboloid (a parabola in Fig. 2) - the flat space. With $t > 0$ $(k < 0)$ the intersections are two-sheet hyperboloids (hyperbolas in Fig. 2) - open spaces of negative curvature.

Fig. 2 faithfully represents not only the topology of the general Stephani solution, but also several details of its geometry - more than would be worth discussing in this place (see Ref. 20). Only the singularity at $x = y = z = 0$ seen on Fig. 2 looks differently in the general case. It is then a true curvature singularity, and it occurs at different values of (x, y, z) for every t. It is an additional singularity to the one predicted by the Hawking-Penrose theorems (21) which occurs also in the FLRW models. The additional singularity can be avoided when the functions k(t) and R(t) and their derivatives obey certain inequalities (20). If $k(t) > 0$ for all t, then the inequalities can be readily fulfilled. Otherwise, they imply that pressure must be negative somewhere. This, in turn, can only be avoided by matching the Stephani Universe to an empty space solution. In any case, however, the weak energy conditions of Hawking and Ellis (21), $\varepsilon \ge 0$ and $\varepsilon + p \ge 0$, can be fulfilled.

7. IN WHAT SENSE IS THIS UNIVERSE HOMOGENEOUS?

The pressure and acceleration scalar do depend here on spatial coordinates. On the other hand, we assumed in sec. 3 that all the 3-spaces t = const should be intrinsically homogeneous. Is this a contradiction?

No - because pressure and acceleration are not intrinsic properties of these 3-spaces. They are fields defined on the 4-dimensional spacetime (or on 4-dimensional subsets thereof). As such, they have well defined values over the spaces t = const. These values, however, can never be calculated if we are given only the geometry of a single 3-space t = const - they are determined by the whole 4-dimensional metric tensor through the Einstein´s field equations. The theory of relativity is telling us here, in its own language, the message known from statistical physics: it is impossible to determine pressure (in any kind of matter) by an instantaneous measurement. The measurement must always take a finite time (the pressure must be defined over a continuous family of t = const spaces), and only afterwards can we determine momentary values of pressure - as limits at $\Delta t \to 0$ of mean values over time-intervals Δt. This fits with the microscopic definition of pressure - as the mean momentum transferred by the gas particles to a unit surface in a unit of time.

Let us consider a more general spacetime in which the 3-spaces t = const are orthogonal to the t-lines, but have arbitrary intrinsic geometries:

$$ds^2 = D^2 dt^2 - h_{ij} dx^i dx^j , \tag{7.1}$$

where i, j = 1, 2, 3 and all the functions (D, h_{ij}) are arbitrary. Let us assume this metric fulfills the Einstein´s field equations with a perfect fluid source whose velocity field is tangent to the t-lines. The density of matter can then be calculated to be

$$\kappa\varepsilon = R(h)/2 + \{(h^{ij} h_{ij,t})^2 + h^{ij}{}_{,t}\, h_{ij,t}\}/8D^2 \tag{7.2}$$

where R(h) is the 3-dimensional scalar curvature of the metric h_{ij}. Eq. (7.2) shows that also matter density need not be spatially homogeneous when h_{ij} is. It happens to be so for the Stephani Universe by accident (and for the Bianchi type models by assumption).

8. IS THE STEPHANI MODEL COMPATIBLE WITH OBSERVATIONS?

Since the FLRW models are contained in this one as special cases, and are themselves believed to be good models of the observed Universe, the answer to the question asked above is immediate: yes, the functions

$k_0(t)$, $x_0(t)$, $y_0(t)$ and $z_0(t)$ can always be chosen to vary so slowly that no observation can distinguish them from constants to which they reduce in the FLRW limit. This statement raises a further question: what are the limits imposed by observations on the derivatives of these functions? This will be a subject of a separate study. An ultimate question is however: can the Stephani model describe anything that the FLRW models could not? The calculations in it are undoubtedly more involved, so does it pay off to use it?

To the author, it was interesting to learn that the classification of cosmological models into the open, the flat and the closed one is not required by Einstein's theory itself, but is an artifact of the very strong symmetry requirements imposed on the FLRW models a priori. The Stephani model had thus at least this conceptual advantage. Whether it has any others, remains to be seen. Further generalizations are needed in any case, since, with spatially homogeneous matter-density, the model cannot serve to describe the galaxy formation in a nonperturbative manner.

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