IRREGULAR COSMOLOGICAL MODELS

Rozprawa habilitacyjna

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1. Why should irregular cosmological models be considered at all?

The foundation of the present-day theoretical cosmology are the well known Friedman - Lemaitre - Robertson - Walker (FLRW) solutions of the Einstein's field equations (1 - 4). They were derived from the Einstein's theory under the very strong assumptions that the spacetime is homogeneous and isotropic. These assumptions were not meant to reflect our knowledge about the Universe, but rather our ignorance: at that time (1930-ies) no structures larger than galaxies were known. The homogeneous and isotropic distribution of galaxies was thus a reasonable first hypothesis which at the same time made the Einstein's equations tractable.

The models proved very successful in describing several observable properties of the Universe, like the Hubble's expansion law (5), the abundance of helium (6) or the microwave background radiation (7 - 8). These successes are often understood as the confirmation of the assumptions underlying the FLRW models. In fact, they only confirm that the Universe was much hotter and denser in the past than it is now and that it <u>was</u> very nearly isotropic at the time when the radiation last interacted with massive particles. Even within these classical models massive particles and radiation are considered as two independent components of matter which decoupled in the moment of last scattering (about $5\ 10^5$ years after the "Big Bang" (9)) and later evolved independently. Whatever happened to particles afterwards, did not affect the distribution of radiation. Therefore, even the homogeneity and isotropy of the radiation do not force upon us a model in which <u>all matter</u> is distributed so symmetrically. Both observations and theoretical considerations show that there is enough room for more general cosmological models to be accomodated in the existing observational data. Let us consider observations first.

The Hubble parameter was originally estimated to be about 500 km/sec.Mpc (10). Subsequently, this estimate was changed several times, mostly downwards (today this value is believed to lie between 50 and 100 km/sec.Mpc (11)). The reason of these revisions was nearly always the same: new discoveries (of intervening intergalactic matter, evolutionary effects, etc.) were showing that the distances to other galaxies were calculated incorrectly. These distances are, however, the fundamental data for calculating the spatial distribution of galaxies. So many revisions prove that the methods of distance measurement were (and probably are) far from satisfactory, and so today´s data must be taken with great caution, even if their <u>formal</u> errors are claimed small. Moreover, the data assumed reliable, today´s state of the art is as follows: the galaxies are grouped into clusters and shells surrounding large voids which contain no visible matter at all (12) (This conclusion was reached simply by estimating the distances to galaxies with Hubble´s expansion law, i.e. by a more precise analysis of the same data that were earlier used to support the claim of Universe´s homogeneity on a smaller scale - a further hint that such claims were insufficiently justified). In spite of that, it is sometimes further claimed that the Universe is homogeneous - on a still larger scale (13). Such a large scale homogeneity coupled with small scale inhomogeneity is not properly described by a spacetime with a continuous transitive group of symmetry (curve a in Fig. 1) of which the FLRW spacetimes are examples. A more appropriate description would be a spacetime with a <u>discrete</u> group of symmetry in which the matter density would be given by a function like curve b in Fig. 1 (see similar remarks by Ellis (19)). Such distribution of matter does not distinguish any single observer because, if the space is infinite, there exist infinitely many identical copies of any chosen finite portion of matter distributed regularly in space. Such a solution can only be found, however, if the assumption of continuous homogeneity is relaxed altogether.

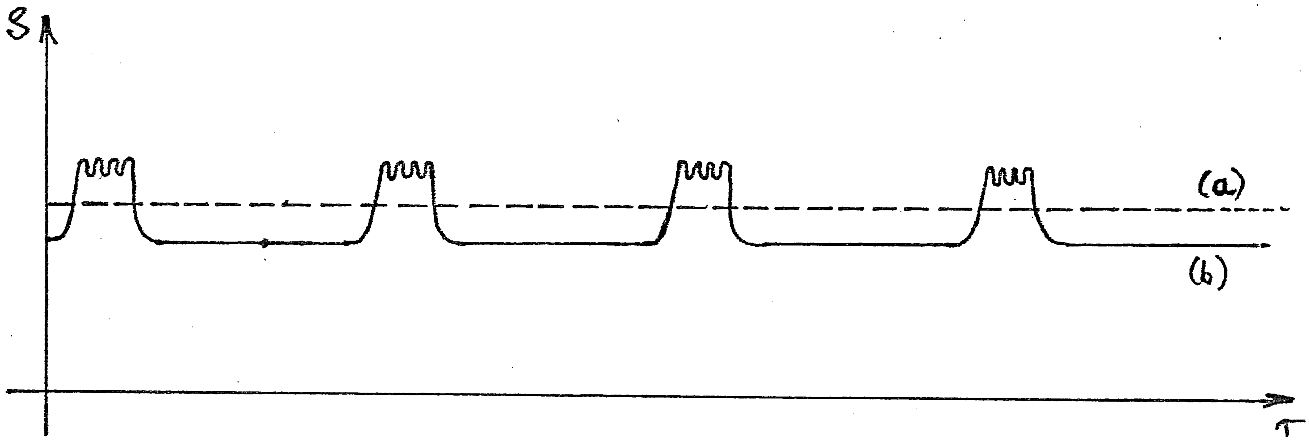

Fig. 1. Matter density vs position in a 3-space t = const of a spacetime that is homogeneous with respect to (a) a continuous group of symmetry, (b) a discrrete group of symmetry.

It should be stressed at this point that the copernician principle only requires that no place in the Universe be preferred. It does not say that all the places in the Universe should be exactly identical. The latter assumption fulfills the former, but is much stronger.

So much can be said basing on the observations of luminous matter. The dynamics of galaxies and of clusters of galaxies give a lot of evidence that there exists a large amount of matter in the Universe which does not emit any kind of observable radiation, i.e. is not directly visible at all (see e.g. (14)). No estimates were made of the density distribution of that dark matter and none seem at present to be possible. How can one claim that something we do not see has a constant density? - is a question that should be answered by the proponents of homogeneous models of the Universe.

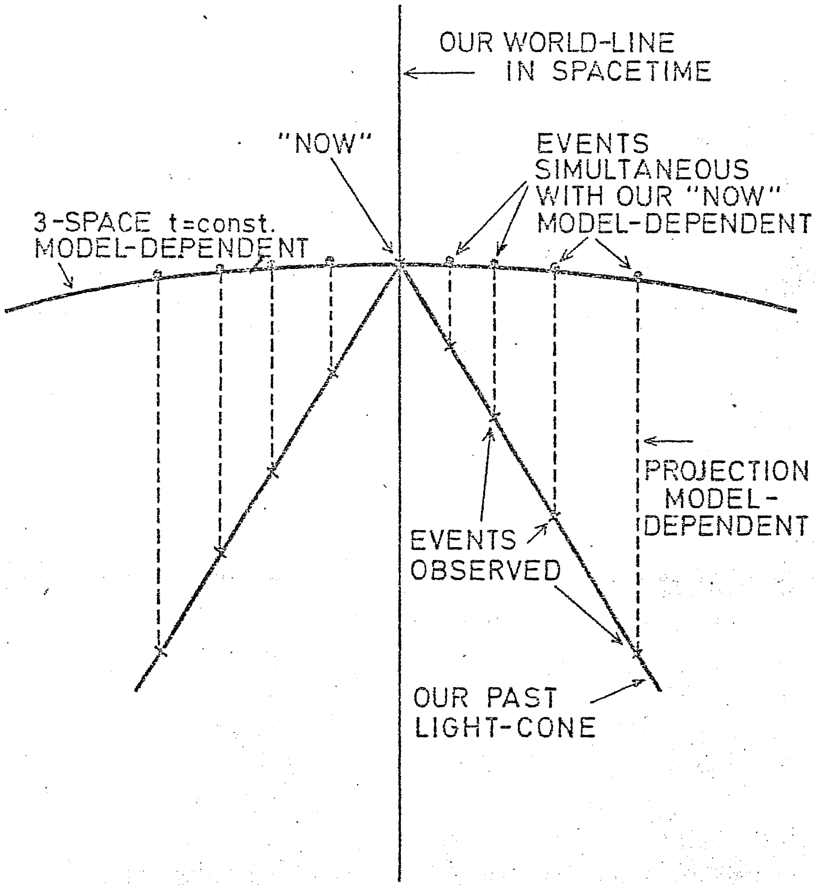

Fig. 2. The spatial distribution of matter in the Universe is deduced from observations through model-dependent procedures. See explanation in the text.

The foregoing arguments were based on observational data. A purely theoretical argument also shows that more general models of the Universe can be reconciled with the existing data (Fig. 2, after Ref. 15). Only the events lying on our past light cone are directly observed - and only the directions to them can be measured with a satisfactory precision, their positions along the directions cannot (because the distance estimates rely on risky assumptions, e.g. that the absolute brightness of the n-th brightest galaxy in a cluster is the same in each cluster (16)). All the other data that are needed to calculate the spatial distribution of matter are inferred therefrom through model-dependent procedures: 1. Through each event on the light cone we draw a world-line representing the history of that portion of matter, e.g. a galaxy (the equations of those lines can only be calculated given a specific class of spacetimes); 2. Through the event "now" on our world-line (the vertex of the light cone) we draw a hypersurface S of events simultaneous with our present event (Even within a fixed model this depends on the reference system chosen. The reference system is usually attached to a physical structure in the spacetime, e.g. the congruence of matter world-lines); 3. The points of intersection of the world-lines with the hypersurface S represent the present positions of the galaxies (The positions of those points on S depend on the slopes of the matter world-lines, i.e. on the velocity of expansion, given the model and given S. This velocity can be calculated from the observed redshift - provided we know precisely what part of the redshift is of cosmological origin). 4. Only now can we calculate the spatial distribution of matter. It certainly does not help the case that astronomers ignore these theoretical subtleties and use simply the Euclidean space as a background for interpreting the observations. A positive program for observational cosmology which would consistently use only the language of general relativity to relate the observations to the calculated properties of theoretical models has recently been elaborated by Ellis and coworkers (17 - 19).

The foregoing discussion should have shown that generalizations of the FLRW models are <u>possible</u> given the existing data. There is at least one instance in which a generalization is seen to be <u>necessary,</u> and that is the formation of galaxies. Taken literally, the FLRW models tell us quite simply that no galaxies may ever have formed out of a homogeneous and isotropic background. All theories of galaxy formation must consider perturbations of the FLRW models where the initial inhomogeneities either appear as statistical fluctuations of density or are introduced quite arbitrarily (see e.g. (20)). If we have to, as we do, go beyond the FLRW models, it is at least equally reasonable to do it by considering exact generalizations instead of approximate perturbations.

The author of this paper is in a definite minority, but by no means alone with his criticism of "standard cosmology". Similar concerns were expressed by Ellis (17 - 19, 21 - 23), MacCallum (24) and Mashhoon (25).

In addition to inhomogeneities, the FLRW models ignore one more feature of the real Universe, namely the rotation of matter. Already in 1946 Gamow (26) put forward the following question: All galaxies and all stars rotate. The stars do rotate because they condensed out of a rotating galaxy. Why do galaxies rotate? Perhaps because they condensed out of a rotating Universe.

Upper limits on the large scale rotation of the Universe were found by Hawking (27) to be rather small, lying below 10^{-14} rad/yr. However, postitive evidence of rotation of the order 10^{-13} rad/yr seems to have been found recently (28). This effect can only be described in a cosmological model more general than the FLRW ones.

Since the FLRW models proved so successful in so many respects, the more general new models should contain them as special cases, i.e. as first approximations. The treatise herewith presented is concerned with generalizations of the FLRW models fulfilling this last condition. It consists of 6 parts:

1. "Cylindrical rotating Universe" (29, Paper I).
2. "All flow-stationary cylindrically symmetric solutions of the Einstein´s field equations for a rotating isentropic perfect fluid" (30, Paper II).
3. "Ellipsoidal spacetimes, sources for the Kerr metric" (31, Paper III).
4. "A Newtonian model of the source of the Kerr metric" (32, Paper IV).
5. "Spacetimes with spherically symmetric hypersufaces" (33, Paper V).
6. "On the global geometry of the Stephani Universe" (34, Paper VI).

Although no definite solution to the problem of inhomogeneities or that of rotation was found, a formalism for investigating rotating cosmological models was introduced, a fruitful excursion into the theory of stationary axisymmetric gravitational fields was made, and an interesting generalization of the FLRW models was rediscovered and investigated in the cosmological context. The six parts of the treatise are discussed in more detail in the sections that follow.

2. Cosmological models with rotation (Paper I).

Several solutions of the Einstein´s field equations are known in which the material source is a rotating perfect fluid or dust. Nearly all of them are, however, stationary, i.e. the matter does not expand. For this reason they are not interesting from the point of view of cosmology. At the time when Paper I (29) was being written, no solution with both expansion and rotation was known. In the meantime one appeared (35), but it does not contain the FLRW models: in the limit of zero rotation it reduces to a stationary spacetime. The aim of Paper I remains thus unfulfilled until today: no physically interesting model of the Universe with rotation exists as a complete solution of the Einsteins equations.

In Paper I an attempt was undertaken to obtain such a model by relaxing, to the minimal degree possible, the assumptions that lead to the FLRW models. The FLRW models are spherically symmetric. In the presence of rotation the spherical symmetry must disappear since the vorticity vector defines a preferred direction in every point. The axial symmetry around the vorticity vector is the most that can remain of spherical symmetry in this case. Homogeneity along the vorticity vector may be assumed as a remnant of the 3-dimensional homogeneity of the FLRW spacetimes. As a starting hypothesis it was assumed that the generators of these two symmetries commute and are orthogonal.

If we would now assume homogeneity also in the directions perpendicular to the vorticity vector, then, together with the axial symmetry, we would obtain a spatially homogeneous model of one of the Bianchi types. These are, however, not very well suited to describe a rotating Universe. In the Bianchi models, the t-coordinate lines are orthogonal to the homogeneous spaces t = const and so have no rotation. The world-lines of rotating matter cannot thus coincide with these t-lines. Hence, in a rotating Bianchi-type model the homogeneous hypersurfaces are an artificial structure which is independent of matter that exists in the Universe: the observers comoving with matter would not see their local rest-spaces to be homogeneous. Therefore it seemed more reasonable to abandon homogeneity altogether.

Since the assumptions described above were more general than those underlying the Bianchi-type models, the latter might be contained there as special cases. In that case, the symmetry group assumed (two dimensional commutative) should be a subgroup of a Bianchi-type group. All Bianchi types except VIII and IX do contain two dimensional abelian subgroups (36). Thus type IX is excluded from this consideration, and together with it the closed Friedman - Lemaitre models.

The coordinate system used in Paper I was introduced by Plebański (37) and derived in more detail in the author's PhD Thesis (38, see also 39). In short, it is a coordinate system in which the equations of motion and the equation of continuity for the rest-mass density of a perfect fluid are identically fulfilled. These coordinates can be introduced whenever the motion is isentropic (entropy per particle = const) and rotational (otherwise two of the coordinates are undetermined). They have invariant geometrical meaning (see (38)), but are most easily defined by the statement that the vector fields of the fluid velocity u, vorticity w and the metric tensor $g_{\alpha\beta}$ assume the form:

$$
\begin{aligned}
&u^{\alpha} = H\delta^{\alpha}_{0}, \\
&w^{\alpha} = \rho H^{-1}\delta^{\alpha}_{3}, \\
&g_{00} = H^{-2}, \\
&g_{01} = x^{2}H^{-2}, \qquad\qquad (2.1) \\
&g_{02} = g_{03} = 0, \\
&\det(g_{\alpha\beta}) = -\rho^{-2}H^{-2},
\end{aligned}
$$

where ρ is the density of the rest-mass of the fluid and H is the enthalpy per particle:

$$H = (\varepsilon + p)/\rho c^2 . \tag{2.2}$$

The coordinates (x^0, x^1, x^2, x^3) are defined up to the transformations:

$$
\begin{aligned}
x^0 &= x^{0'} - S(x^{1'}, x^{2'}),\\
x^1 &= F(x^{1'}, x^{2'}),\\
x^2 &= G(x^{1'}, x^{2'}),\\
x^3 &= x^{3'} + T(x^{1'}, x^{2'}).
\end{aligned}
\tag{2.3}
$$

where F and G must obey the equation:

$$F,_{1'} G,_{2'} - F,_{2'} G,_{1'} = 1, \tag{2.4}$$

and S is then determined by:

$$S,_{1'} = GF,_{1'} - x^{2'} , \quad S,_{2'} = GF,_{2'} . \tag{2.5}$$

The assumptions about symmetry made in Paper I lead to the conclusion that the metric in this coordinate system depends only on x^1 and x^2 and has the further property:

$$g_{13} = 0. \tag{2.6}$$

The field equations yield then the further result: g_{23}/g_{33} is a function of x^2 only. Thus by applying the coordinate transformation (2.3) with $T(x^2) = -\int (g_{23}/g_{33})\, dx^2$ we obtain:

$$g_{23} = 0. \tag{2.7}$$

A more detailed version of Paper I was intended to be published elsewhere, but after being turned down at the Proceedings of the Royal Society and lost at the International Journal of Theoretical Physics it remained in the preprint form only (40). It contains all the calculations missing in Paper I (29) and can be obtained from the author on request. A further integral of the field equations was obtained there, but is too complicated to be worth quoting.

3. Stationary cylindrically symmetric solutions of the Einstein´s equations with a perfect fluid source (Paper II).

The method used in Paper I appeared well suited for stationary cylindrically symmetric solutions of the Einstein´s field equations. The author derived and investigated a large class of such solutions in his PhD Thesis (38, 41). They were obtained under the assumptions that there exists a Killing vector colinear with the velocity field of matter and another one colinear with the vector field of rotation in a spacetime filled with a rotating perfect fluid. The field equations required then the existence of a third Killing field, the second and the third Killing fields corresponding to cylindrical symmetry. However, the solutions obtained were, in a way, incomplete since they determined simultaneously the matter density and the pressure and thus also the equation of state. This was contrary to intuition, since one normally expects that an arbitrary equation of state can be imposed onto a solution as an additional feature, not determined by the field equations. Paper II (30) explained what happened in (38) and (41): a specific proportionality factor was assumed there between the second Killing field and the rotation vector what limited the generality of the result. The solutions found in Paper II were obtained by the method of Paper I and of (38 - 41) from the following assumptions:

1. The velocity field of matter is colinear with a Killing vector field, the proportionality factor being an arbitrary function.

2. The vector field of rotation is proportional to another Killing vector field and the proportionality factor is another arbitrary function.

3. A third Killing vector field also exists which corresponds to axial symmetry. Its form was initially assumed to be quite arbitrary.

No commutation relations were assumed between the Killing vectors. In spite of that, the solutions obtained cannot be called the most general cylindrically symmetric solutions, just because of the alignments between the Killing vectors and the vector fields characterising the fluid flow that were assumed. Although these assumptions are quite natural, it should still be investigated what happens when the timelike Killing field is not colinear with the fluid flow and when no spacelike Killing field coincides with the rotation vector.

The following results were obtained in Paper II:

1. All the three Killing vector fields must commute unless the third one is a linear combination of the first two (in which case the spacetime has a smaller symmetry group than was assumed). Thus the metric depends on only one coordinate x^2.

2. The proportionality factor between the flow velocity and the timelike Killing field must be exactly as assumed in (38).

3. The proportionality factor between the second Killing field and the vector of rotation contains an arbitrary function $f(x^2)$.

4. All components of the metric are algebraically determined through the solutions of an ordinary linear homogeneous equation of second order ((4.10) in Paper II) whose coefficients depend on the arbitrary function $f(x^2)$. For this reason the most general solution cannot be found in a closed form.

5. With $f(x^2)$ present both in the matter density $\rho(x^2)$ and in the pressure $p(x^2)$ the equation of state is completely arbitrary.

4. Ellipsoidal spacetimes (Paper III).

It followed from Paper I that the method discussed there could not incorporate the closed FLRW models. In order to find a cosmological model with rotating matter which would reduce to a closed FLRW model in the limit of vanishing rotation, an independent method had to be created.

A natural first idea was the following. In the FLRW models spheres are geometrically distinguished as orbits of the symmetry group. In the presence of rotation, spheres should deform into something axially symmetric. The simplest deformation of a sphere is a spheroid. An appropriate generalization of the FLRW models would then be a spacetime in which the spheroids are geometrically preferred structures.

The problem was mathematically intractable in this generality, but, similarly as in Paper I the approach appeared useful in the stationary case: The spacetime metric thus obtained was strikingly similar to the Kerr solution (42) in the Boyer-Lindquist coordinates (43). This observation gave rise to Paper III (31). It appeared that the Kerr metric can be constructed from confocal spheroids through the following procedure:

1. In the 3-dimensional Euclidean space the oblate spheroidal coordinates (r, θ, ϕ) are introduced in which the r = const surfaces are confocal spheroids, the θ = const surfaces are one-sheet hyperboloids confocal to the spheroids and ϕ is the azimuthal angle. From there, the metric form of a spheroid is read out.

2. A curved 3-dimensional Riemannian space (with positive-definite metric) is constructed out of the spheroids such that the spheroids are the surfaces r = const in it, and the r-lines are orthogonal to them.

3. A 4-dimensional spacetime with Lorentz signature is constructed from the spaces of point 2 and from a congruence of

timelike lines in such a way that the spaces are locally orthogonal to the lines, i.e. the metric tensor of the 3-spaces, $h_{\alpha\beta}$, is obtained form the spacetime metric $g_{\alpha\beta}$ through the projection:

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta \tag{4.1}$$

where u is the vector field tangent to the timelike congruence. With a certain choice of u and of the curvature of $h_{\alpha\beta}$, the Kerr metric emerges (in a coordinate system related to that of Boyer and Lindquist by a linear coordinate transformation). The spacetimes of this class were named ellipsoidal.

It was hoped that a source of the Kerr metric could possibly be found in the same class of spacetimes. The idea was as follows: If one of the ellipsoids (given by the equation r = const) should be an outer surface of the source, then the θ -dependence of both the exterior and of the interior metric on that surface should be the same. Perhaps, then, both metrics depend on θ in the same way for every value of r. To check this guess, a trial metric was substituted into the Einstein´s field equations with a perfect fluid source. The metric was obtained from the Kerr metric in the ellipsoidal form in such a way that a different function of r was substituted for every constant of the Kerr metric. The field equations were then treated as identities in θ. Altogether, a set of about 60 ordinary differential equations for 9 unknown functions of r resulted. It appeared that the only solution of these equations was the Kerr metric itself. After Paper III was submitted for publication, Roos (44) proved a similar result. By analysing the general set of Einstein´s equations for a stationary-axisymmetric metric with the boundary condition that the interior metric matches continuously to the Kerr metric across a hypersurface S he showed that S cannot coincide with any of the ellipsoids r = const.

In Paper III a survey of literature on the sources of the Kerr metric was also made. Judging from later citations, this part of Paper III was the most useful one for readers. The survey included all papers about any kind of sources for the Kerr metric written before the end of the year 1975. The various methods of approach were classified and compared - and it was shown in a few instances that an excessive importance was assigned to conclusions from oversimplified assumptions or methods.

It is worth mentioning that the extreme difficulty of calculations made for this paper prompted the author to start a joint work with Dr. Marek Perkowski form the Warsaw Technical University on a computer program for algebraic calculations in general relativity. The program was written in the years 1977 - 1979 and since 1979 is available for users (45). Afterwards, it was extended for other specialized programs and implemented on several types of computers (46) and is slowly gaining an international reputation.

5. Newtonian model of an ellipsoidal spacetime (Paper IV).

The ellipsoids of which the Kerr spacetime was shown to consist (in Paper III) reminded of the equipotential surfaces from Newtonian theory. Paper III contains a few citations from the literature where a connection between the Kerr solution and Newtonian fields which are constant on ellipsoids was suggested by different methods. It was therefore interesting to see what kind of body could produce such a field in Newtonian physics. To be sure: the Newtonian field which results from the Kerr metric through the conventional "Newtonian limit" procedure is not constant on ellipsoids, it is given by the potential:

$$V_K = \frac{2mr}{r^2 + a^2 \cos^2\theta} . \qquad (5.1)$$

Paper IV (32) was devoted to the (purely Newtonian) investigation of a gravitational field whose equipotential surfaces are confocal spheroids, and of a source of this field. One source was known since 1840: Chasles (47) found that such a field is produced by an infinitely thin spheroidal shell of finite mass which coincides with one of the equipotential surfaces and has a constant potential inside. Such a source seemed very artificial, however, from the astrophysical point of view. In Paper IV a source was found which is a finite portion of a rotating perfect fluid in which the density, pressure and the scalar of rotation are distributed continuously throughout the interior. Unfortunately, another undesirable feature appeared in the source: the density of mass either is infinite on the focal ring of the ellipsoids or is zero on the disk spanned on this ring and attains its maximum somewhere outside the disk. This unusual behavior might have resulted from a simplifying assumption that the equipotential surfaces inside the body are confocal ellipsoids of the same family as those outside the body. This assumption is not necessary, and it still remains to be verified whether a nonsingular source exists.

A ring singularity is familiar from the investigation of the Kerr metric and its possible sources. It is interesting that it appears only in the interior of the source: the exterior field remains nonsingular everywhere even if the body generating the field is squeezed to the focal disk of the ellipsoids.

Particularly appropriate to the investigations of the Kerr metric is the following result of Paper IV: the exterior gravitational field has only one parameter which describes its nonsphericity, a, and so all the multipole moments are algebraically dependent: after one of them is fixed, all the others are determined. This holds also for the Kerr metric, and this statement was sometimes used as an argument that the (still unknown) material source must have a very peculiar, rigidly determined structure. The solution of Paper IV seems to be a counterexample: the density distribution contains an arbitrary function f(r) and so is fairly general.

Further study of the connection between the solution of Paper IV and the theory of relativity is under way (48).

6. Spacetimes of intrinsic spherical symmetry (Paper V).

The considerations of section 1, and in particular Fig. 2, show that what is checked against astronomical observations is in fact the 3-dimensional space t = now rather than the whole spacetime. It is then a natural question, to what extent the 3-geometries of the spaces t = const determine the 4-geometry of our spacetime.

To answer this question, a technique of "building" the spacetime of a set of its subspaces was applied, similar to the one used in Paper III. At the time when Paper V (33) was being written, the proposal of Collins (49) was published to investigate the so called "intrinsically symmetric" spacetimes in which symmetry groups operate on families of subspaces rather than on the whole space. Thus by accident Paper V appeared to be one of the first realisations of this program. The real inspiration to write this paper came, however, from the publications of Ellis and coworkers (17, 18, 21 - 23).

In the beginning it was assumed that:

1. The spacetime is a congruence of 3-spaces, each of which is spherically symmetric.

2. The spaces are orthogonal to the family of t-coordinate lines and are given by the equations t = const.

3. The whole spacetime not necessarily has any symmetry.

4. The Einstein's field equations are fulfilled, the source being either a perfect fluid of dust or the Λ-term; the empty space equations were also included.

5. The matter in the spacetime, if there is any, either moves along the t-lines of deviates from them only in radial directions (in order to make the lack of spherical symmetry in the spacetime not too easily visible).

The field equations appeared to impose very strong limitations on the solutions because most of the separate cases that had to be considered led to spacetimes that were spherically symmetric in the ordinary sense. All the dust and empty space solutions, both with and without the Λ-term, belong to this class. With a perfect fluid source, however, there exists one solution which has in general no symmetry at all. Its metric form is:

$$ds^2 = D^2 dt^2 - (1 + Kr^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (6.1)$$

where K = const,

$$D = r\{A(t) \sin\theta \cos\phi + B(t) \sin\theta \sin\phi + C(t) \cos\theta\} + E(t)(1 + Kr^2)^{1/2} + s, \qquad (6.2)$$

A, B, C, E are arbitrary functions of t, and either s = 0 or s can be scaled to 1 by a coordinate transformation. The solution is conformally flat, the energy density and pressure are given by:

$$\kappa\varepsilon = -3K = \text{const}$$
$$\kappa p = 3K(1 - 2s/3D), \qquad (6.3)$$

$\kappa = 8\pi G/c^4$. The solution was first found by Stephani (50) and is a simultaneous generalization of the interior Schwarzschild solution (which results from it when A = B = C = 0, E = const) and of the deSitter solution (which results when s = 0). Although it does not seem to be of any astrophysical interest, it shows that such solutions do indeed exist in which preferred sets of of hypersurfaces have a larger symmetry group than the whole spacetime. Paper V contains a short investigation of the properties of the solution (6.1-3). It appears that a spherically symmetric spacetime results when the t-lines are geodesics, and a spacetime without a symmetry results if the lines are nongeodesic.

In the second part of Paper V a similar study was undertaken of spacetimes which are "intrinsically" homogeneous and isotropic in the sense of Collins (49), i.e. are congruences of spaces of constant curvature. Similar assumptions were made. Assumption 1 was modified to:

1´. Each spatial section is a space of constant curvature.

while the Λ-term and the empty space were dropped from assumption 4 since they did not promise anything interesting to emerge.

In this case, an interesting generalisation of the FLRW models appears which was also first found by Stephani (50), but outside the cosmological context. The solution is given by the formulae:

$$ds^2 = D^2 dt^2 - (R^2/V^2)(dx^2 + dy^2 + dz^2), \qquad (6.4)$$

$$D = F - \frac{R}{V}\frac{\partial}{\partial t}\left(\frac{V}{R}\right), \qquad (6.5)$$

$$V = 1 + \frac{1}{4} k \{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\}, \tag{6.6}$$

$$k = (C^2 - 1/F^2) R^2, \tag{6.7}$$

$$\kappa\varepsilon = 3C^2, \tag{6.8}$$

$$\kappa p = -3C^2 + 2C \frac{dC}{dt} \frac{V}{R} / \frac{\partial}{\partial t}\left(\frac{V}{R}\right), \tag{6.9}$$

$$\theta = -3/F, \tag{6.10}$$

$$u^\alpha = D^{-1} \delta^\alpha_0, \tag{6.11}$$

$$a^0 = 0,$$

$$a^i = (V^2/DR^2) D_{,i}, \qquad i = 1, 2, 3. \tag{6.12}$$

where F, R, C, x_0, y_0, z_0 are arbitrary functions of time, ε is the energy density, p is the pressure, θ is the scalar of expansion of the fluid flow, u is the 4-velocity of the perfect fluid source and a is the acceleration field. The shear and rotation of the fluid are zero.

The solution has in general no symmetry at all. Its most striking property is the fact that k is a function of t and its sign is not determined. Since k is the curvature index of the 3-spaces t = const, one sees that in this spacetime some spacelike sections have positive curvature (and so should be closed) while some others have negative or zero curvature (and so should be open). Other differences with the FLRW solutions are the following:

1. Matter moves with acceleration, i.e. not on geodesic lines.

2. The equation of state is not of the form $\varepsilon = \varepsilon(p)$, but depends on the position in the space: $p = p(\varepsilon, x, y, z)$.
This last property means that a single thermodynamic function of state (e.g. pressure) does not suffice to describe matter in this model, at least one other function is necessary, e.g. temperature which would have different values in different places. This situation is thus even more realistic than in the FLRW models.

The investigation of global geometrical properties of this Stephani Universe was postponed to Paper VI. In Paper V only local properties and the derivation of the solution were discussed.

The Stephani Universe reduces to a FLRW model when any one of the following situations occurs:

1. The functions x_0, y_0, z_0 and k are all constant.

2. The acceleration field vanishes (i.e. matter moves on geodesics).

3. The equation of state is of the form $\varepsilon = \varepsilon(p)$, i.e. it does not depend on position.

This solution is conformally flat, and moreover it is the most general conformally flat solution with a perfect fluid source and nonvanishing expansion (51).

In this model, matter flows along the t-lines. The derivation in Paper V shows that a generalization should be expected when this assumption is dropped. A further generalization would be possible in which the t-lines are not orthogonal to the spaces t = const. This still awaits to be done.

It is seen from the derivation in Paper V that the resulting intrinsically homogeneous spacetime depends on the coordinate representation of the sections. Namely, the spacetime metric is obtained on replacing constants in the 3-space metric by arbitrary functions of t. Coordinate transformations in the 3-spaces can generate spurious constants which have no meaning in the intrinsic geometry of the spaces. The functions that replace them are, however, meaningful: they describe the way in which the spaces are stacked together to form a spacetime. In particular, when $x_0 = y_0 = z_0 = \text{const}$, the coordinate centers of spherical symmetry in the spaces $t = \mathrm{const}$ are all placed on the same t-line $x = x_0$, $y = y_0$, $z = z_0$, and so the resulting spacetime is spherically symmetric, too. With arbitrary x_0, y_0 and z_0, the centers of symmetry in the 3-spaces are arbitrarily shifted with respect to the t-lines and no symmetry in the spacetime remains.

The results of Paper V were obtained with the help of the program ORTOCARTAN (45, 46, 48) which was also used for Paper VI.

7. Global properties of the Stephani Universe (Paper VI).

To most of the readers and auditors which first heard of the Stephani Universe, and to the author himself, it was difficult to imagine how the same spacetime could have some spacelike sections closed and some others open. Therefore, a separate study was devoted to the global geometrical picture of this solution. It resulted in Paper VI (34).

It was shown by Stephani (50) that this solution can be embedded in a flat five-dimensional space. He only proved, however, the existence of the embedding. To construct it explicitly in the most general case was an impossible task because the solution contains 6 arbitrary functions of time. The embedding would thus have to be defined through solutions of ordinary differential equations containing arbitrary functions as coefficients, or through indefinite integrals of these functions. It was more instructive to study a special case in which the embedding could be performed explicitly.

Such a special case results when $C = \mathrm{const}$; the Stephani Universe reduces then to the deSitter solution. It was further assumed $x_0 = y_0 = z_0 = 0$, $R = \mathrm{const}$, $k = -t$. In the special case $C = \mathrm{const}$ these additional assumptions amount just to a choice of a simpler coordinate system (foliation) because once $C = \mathrm{const}$, the solution (6.4-12) describes the deSitter spacetime irrespectively of the forms of the other functions.

The deSitter manifold is then described as a 4-dimensional one-sheet hyperboloid embedded in a 5-dimensional pseudoeuclidean space. The discussion in Paper VI and the figures are limited to the intersection of this 4-dimensional hyperboloid with the subspace $\{\theta = \text{const}, \ \phi = \text{const}\}$, the resulting hyperboloid being 2-dimensional, but this case carries all the important information. The axis of the hyperboloid is parallel to the Z coordinate axis in the flat space and pierces the (X, Y) plane in the point $\{X = 0, Y = 1/C\}$. The t = const spaces are intersections of the hyperboloid with planes containing the X axis, the t-lines are its intersections with planes containing the Z axis. If a t = const plane is tilted to the (X, Y) plane at a sufficiently small angle, the intersection is an ellipse (i.e. a closed space of positive curvature). If it is tangent to the asymptotic cone of the hyperboloid, the intersection is a parabola - a flat open space. If it is tilted at a still greater angle, the intersection is a hyperbola - an open space of negative curvature. It was noted already long ago by Schrödinger (52) that the de Sitter spacetime may be foliated into spaces of differing topologies, but he did not consider the case when spaces of different topologies belong to the same one foliation.

Next, the generic case was studied in another special case, $x_0 = y_0 = z_0 = 0$, when the spacetime is spherically symmetric, in order to see how faithfully the de Sitter manifold foliated as above represents the general solution. More important of the properties found are:

1. Each section t = const with $k(t) > 0$ is (or can be extended to) a space of finite volume.

2. Each section t = const with $k(t) = 0$ is (or can be extended to) the infinite space R^3.

3. Each section t = const with $k(t) < 0$ is (or can be extended to) a space of infinite volume, but even so extended it will not intersect some of the matter flow lines.

4. Each space t = const with $k(t) < 0$ consists of two disjoint sheets. The one containing r = 0 is called the "near sheet".

5. The infinity of the near sheet of a space with $k(t) < 0$ does not lie on any matter flow line (theorem 5.1).

6. The infinity of a flat space t = const does not lie on any flow line either (theorem 5.2).

7. Whenever k becomes negative, there exist such flow lines on which a finite value of t corresponds either to the infinite future (theorem 5.3) or to the infinite past (theorem 5.4).

The singularity at $r = 0$ seen on the pictures in Paper VI looks differently in the general case. It is then not just a coordinate singularity, but a true curvature singularity, and it occurs at a different value of r for every t. It is an additional singularity to that one which is predicted by the Hawking-Penrose theorems (53) and which occurs also in the FLRW models. The additional singularity can be avoided when the functions $k(t)$ and $R(t)$ and their derivatives obey certain inequalities (section 6 of Paper VI). If $k(t) > 0$ for all t, then the inequalities can be fulfilled without problems. Otherwise, they imply that the pressure must be negative somewhere. This, in turn, can only be avoided by matching the Stephani Universe to an empty space solution.

Some years ago Geroch (54) and Kundt (55) published papers in which they showed that the topology of spacelike sections of a spacetime cannot change from one section to another if certain assumptions are fulfilled. The Stephani Universe does not contradict these theorems because:

1. The Geroch´s theorem says that if the part of spacetime contained between two compact spacelike sections is itself compact and causal, then the two sections are diffeomorphic - and this is fulfilled by the Stephani Universe. Each part of the Stephani Universe which contains an open section simply does not obey the assumptions of Geroch.

2. Kundt showed that if the sections are transverse to a continuous family of timelike directions and the manifold is geodesically time complete, then the sections are either all connected or all nonconnected. In the Stephani manifold all sections are connected.

It was anticipated by Brill (56) and Yodzis (57) that spacetimes with changing topology of spacelike sections might exist, but they did not consider any specific solution. Paper VI is thus the first one which discusses such a change on an explicit example.

8. Problems for the future.

The Stephani Universe of Papers V and VI is still not satisfactory as a cosmological model because the matter density in it depends only on time. Further (or other) generalizations of the FLRW models are thus needed. They can be searched for instance by pursuing the cases left out in the derivation of the Stephani model (e.g. the case when the flow lines do not coincide with the t-lines) or by considering spacetimes with more general geometries of the spacelike sections, e.g. spacetimes with conformally flat sections. The study of the second case is under way (58), but no conclusions came out as yet.

The cosmological predictions of the Stephani Universe should be checked against the existing observational data, in the first step in the traditional framework, in order to see what constraints follow for the functions k, F, R, x_0, y_0, z_0. This would be a good subject for a MSc Thesis. A more ambitious project would be to carry out the same study within the framework proposed by Ellis (17 - 19).

The problem of a realistic cosmological model with rotation is still outstanding.

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