General Relativity and Gravitation, Vol. 15, No. 7, 1983

On the Global Geometry of the Stephani Universe[1]

ANDRZEJ KRASIŃSKI[2]

Max-Planck-Institut für Physik und Astrophysik, Garching b. München, West Germany

Received September 10, 1982

Abstract

A preliminary investigation of global properties of the Stephani solution of the Einstein field equations is presented. This solution generalizes those of Friedman–Robertson–Walker (FRW) in such a way that the spatial curvature index k (a constant in the FRW models) is a function of the time coordinate. The de Sitter solution, which is also a special case of the Stephani solution, is analyzed in the Stephani coordinates to gain insight into the global structure of the manifold and its foliation. The general metric is found to have several properties in common with this example. It has singularities which can be avoided either by matching the solution to an (as yet unknown) empty-space solution or confining the curvature index to be positive at all times.

§(1): *Introduction*

The Stephani Universe is a solution of the Einstein field equations with a perfect fluid source which contains the well-known Friedman–Robertson–Walker (FRW) cosmological models [1, 2] as special cases. As in the FRW models, the hypersurfaces orthogonal to matter world lines have constant curvature, but in contrast to the FRW case the curvature index k is an arbitrary function of the time coordinate, which can change its sign from one hypersurface to another (in the FRW case it was a constant normalizable, if nonzero, to +1 or − 1). Thus, although in each selected moment t = const the 3-space has the same geometry as if it were a subspace of a FRW model, in general the Universe may appear to have a positive spatial curvature at one time and a negative one in another moment.

[1]Work supported by the A. von Humboldt Foundation.
[2]On leave of absence from the N. Copernicus Astronomical Center, Polish Academy of Sciences, Warsaw, Poland.

0001-7701/83/0700-0673$03.00/0

The solution was first found by Stephani [3] as a special example of a space-time embeddable in a flat five-dimensional space, and later reobtained by the present author [4] in a direct search for solutions which are "intrinsically" spherically symmetric in the sense of Collins [5] (i.e., are composed of spherically symmetric subspaces).

The existence of this solution shows that the distinction between the closed and the open Universe is not required by Einstein's theory of gravitation as such, but is due to the very strong symmetry assumptions which are put into the models right from the beginning. Namely, as mentioned in Ref. 4, it is the assumption of homogeneity in the Bianchi sense which results in fixing the spatial curvature index. If the space-time is allowed to have the weaker "intrinsic homogeneity," i.e., to be a one-parameter family of three-dimensional homogeneous spaces (here: spaces of constant curvature) which is not required to have the same symmetry as a whole, then the Stephani Universe emerges as a rather special case.

The difference between these two types of homogeneity can be qualitatively described as follows. We set up observers in all possible spatial locations, let them perform any kind of observations they wish, and ask them to write the chronicles of their findings. In the space-time which is homogeneous in the Bianchi sense all the chronicles should be just identical. In the space-time which is only "intrinsically homogeneous" the chronicles may greatly differ in their descriptions of the details of evolution of the Universe in the vicinity of the observer, but they will still agree with respect to the geometry of each single rest-3-space of matter.

In fact, there were works which aimed, explicitly or implicitly, at such a generalization, but failed to obtain it because of other restrictive assumptions. Namely, Mashhoon and Partovi [6] investigated the uniqueness of the FRW models among spatially isotropic models in which matter moves without shear. They proved these models to follow uniquely if, in addition, the electrical neutrality of matter and an equation of state of the form $\epsilon = \epsilon(p)$ (with $\epsilon > 0$ and $p > 0$) are assumed. They missed the Stephani model just because of the last assumption. Such an equation of state will be shown in Section 2 to be incompatible with this model.

Bergmann [7] assumed the curvature index k to be a function of time, but did not allow for the acceleration field in the fluid source [the function D in equation (1) of this paper was assumed to be equal to 1]. Under these two assumptions the matter cannot be a perfect fluid: there are additional components in the energy-momentum tensor which Bergmann interpreted as heat flow. In the Stephani model matter moves necessarily with acceleration, and this explains why the model did not show up in [7].

Another kind of work in the same spirit is the discussion by H. J. Schmidt [8] of the possible transitions between various Bianchi types of symmetry groups of the spacelike slices in a space-time of an arbitrary symmetry. Several such transitions are allowed if one assumes that the metric is twice continuously differentiable, but their number is limited if one demands in addition that inside the hypersurface on which the Bianchi type actually changes the curvature is bounded, or that the positive energy condition holds. In fact, the transition from type II to type I is excluded by the last assumption, and the author conjectured that this is so for all the other transitions. All this was shown, however, under the assumption that the orthogonal trajectories of the homogeneous slices are timelike geodesics. As will be seen in Section 2, the requirement that the t lines (and so the flow lines of matter) are geodesics reduces the Stephani model to a FRW solution, and is thus a rather strong assumption.

The possibility of the existence of a space-time in which the topology of certain geometrically preferred sections is changing in time has been also anticipated [9-13]. Specifically, it was shown by Geroch [9] that, if a space-time has two compact spatial sections, while its part contained between the sections is itself compact, and causal in addition, then the two sections must be diffeomorphic. Kundt [10] proved that if the space sections are transverse to a continuous family of timelike directions while the manifold is geodesically time complete, then the sections are either all connected or all nonconnected. Yodzis [12, 13] has shown that changes of spatial topology are possible if one slice is obtained from any other by a finite number of spherical modifications in the sense of Morse [12], but did not give any specific example of a space-time with such a topology change.

It is seen from this short overview that no solution qualitatively similar to that of Stephani has so far been considered from a global point of view. This paper discusses the geometry of the Stephani Universe in the vicinity of the spatial hypersurface on which the spatial curvature changes from positive to negative. Section 2 is a review of local properties of the Stephani solution known already before. In Section 3 a special Stephani solution, in which the metric is that of the de Sitter, is discussed in order to display an example of a foliation of a space-time such that each leave has a constant curvature, but the curvature changes sign from one leave to another. In Section 4, the interior geometry of the spatial sections is discussed. In Section 5 it is shown that the general Stephani solution shares several qualitative properties with the de Sitter solution foliated as in Section 3. It is shown that the spatial slices t = const of negative curvature do not intersect those matter lines which are too far from the central line, that if such slices exist then on several matter lines a finite value of the preferred time coordinate corresponds to future timelike infinity, and that if different slices are tangent somewhere, then this leads to singularities. In Section 6 characteristics of the matter congruence are discussed, and it is shown that acceleration and pressure inevitably have singularities unless either the solution is matched to an (as yet unknown) exterior solution or the curvature index is confined to be positive at all times and certain inequalities are imposed on the arbitrary functions and their derivatives. It is concluded in Section 7 that the Stephani model with $k > 0$ is in no point contradictory to the FRW models as concerns the standard observational tests, and was so far refuted only because it did not fulfill the *a priori* philosophical premises which lead to the simplified FRW solutions.

§(2): *Some Properties of the Stephani Universe*

The Stephani solution is given by the following formulas:

$$ds^2 = D^2\, dt^2 - \frac{R^2(t)}{V^2}\,(dx^2 + dy^2 + dz^2) \tag{1}$$

where

$$V = 1 + \tfrac{1}{4} k(t)\,\{[x - x_0(t)]^2 + [y - y_0(t)]^2 + [z - z_0(t)]^2\} \tag{2}$$

$$D = F(t)\left(\frac{\dot{V}}{V} - \frac{\dot{R}}{R}\right) = F\frac{R}{V}\frac{\partial}{\partial t}\frac{V}{R} \tag{3}$$

$$k = [C^2(t) - 1/F^2(t)]\, R^2(t) \tag{4}$$

C, F, R, x_0, y_0, z_0 being arbitrary functions of time. The metric (1) fulfills the Einstein field equations with a perfect fluid source having the energy density

$$\kappa\epsilon = 3C^2(t), \qquad \kappa = 8\pi G/c^4 \tag{5}$$

and the pressure:

$$\kappa p = -3C^2(t) + 2C(t)\frac{dC}{dt}\frac{V}{R}\bigg/\frac{\partial}{\partial t}\frac{V}{R} \tag{6}$$

The metric (1) is conformally flat. This implies that the empty-space solution belonging to the family (1), obtainable as the special case $C = 0$, is just Riemann flat.

The matter source in this solution is a pefect fluid flowing along the t-coordinate lines, having zero shear and rotation, acceleration equal to

$$\dot{u}^i = (V^2/DR^2)\, D_{,i}, \quad i = 1, 2, 3, \;\; x_1 = x_1, \;\; x_2 = y, \;\; x_3 = z, \;\; \dot{u}^0 = 0 \tag{7}$$

and expansion equal to

$$\theta = -3/F \tag{8}$$

The Stephani space-time is locally FRW if and only if one of the following conditions holds:

(a) $k = \text{const}$ and $x_0, y_0, z_0 = \text{const}$
(b) $\dot{u}^\alpha = 0$
(c) $p = f(\epsilon)$, where f is independent of x, y, z

(actually any one of these conditions implies the other two). In general, the Stephani solution has no symmetry. With x_0, y_0, z_0 being constants it becomes spherically symmetric about the single line $x = x_0, y = y_0, z = z_0$, still lacking the homogeneity of the FRW models. Condition (b) means that in the general Stephani model matter flow lines are necessarily nongeodesic; they become geodesics only if the model reduces to a FRW model.

The condition (c) means that the equation of state is in this model position dependent: an equation of the form $\epsilon = \epsilon(p)$ holds along each flow line separately, but is different on each line.

As can be seen from (4), k is an algebraic expression involving three arbitrary functions of time which are not determined by the field equations. The sign of k is not fixed by (4)—it can change with time, too. We shall be concerned with this possibility in what follows. For brevity we will call the change from positive k to negative k the "opening up" of the Universe and denote it $t = t_{\mathrm{op}}$. The opposite change we will call the "closing down" and denote it $t = t_c$.

§(3): *The de Sitter Solution as a Special Case of the Stephani Solution*

It is seen from (5)-(6) and from the vanishing of the Weyl tensor that with C being a constant the Stephani solution reduces to the de Sitter solution (since only C^2 appears in the metric, the case $C^2 < 0$ can also be included in the considerations, so that both de Sitter's solutions are covered). In the coordinate system of (1)-(4) the de Sitter solution is foliated in an exotic way. This foliation, similar to those considered by Schrödinger [14], allows for an explicit display of some of the qualitative features of the Stephani Universe, and so we will consider this special case first.

With C being constant, (1)-(4) gives the de Sitter metric irrespective of the form of the functions $R(t), F(t), x_0(t), y_0(t), z_0(t)$, so that any further assumption concerning the form of these functions amounts to a choice of a special foliation (congruence of world lines). To make the calculations easier we will assume that $x_0 = y_0 = z_0 = 0, R = \text{const}$ and $k(t) = -t$, so that for $t < 0$ the curvature of the spatial sections is positive while at $t = 0$ the Universe "opens up" and remains open for $t > 0$. With this choice of k we consider only the case of a single change of spatial topology, but we will show this at the end of the section to be the only reasonable choice. With the forms of C, R, k, x_0, y_0, z_0 specified above we have

$$F = R/(C^2R^2 + t)^{1/2} \tag{9}$$

$$ds^2 = \frac{R^2 r^4\, dt^2}{16(1 - \frac{1}{4}tr^2)^2\,(C^2R^2 + t)} - \frac{R^2}{(1 - \frac{1}{4}tr^2)^2}\,[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{10}$$

where we have introduced spherical coordinates in an obvious way. Moreover, to make an explicit investigation of geometry possible, we shall limit ourselves to a two-dimensional subspace of (10) given by $\theta = \text{const}$, $\phi = \text{const}$. We see that the metric form (10) is well defined in the coordinate ranges $\{-C^2R^2 < t \leqslant 0, 0 < |r| < \infty\}$ and $\{t > 0, 0 < |r| < 2/\sqrt{t}, 2/\sqrt{t} < |r| < \infty\}$. At $t = -C^2R^2$, $r = 0$ and $r = 2/\sqrt{t}$ the metric becomes singular, but the singularities will be seen later to be spurious ones, due to the coordinate system chosen.[3] We let r assume negative values in the two-dimensional space under consideration, with the convention that the point $(-r, \theta, \phi)$ is identical with $(r, \pi - \theta, \phi + \pi)$ throughout the whole space.

Such a two-dimensional space can be parametrically represented as a hypersurface in the three-dimensional flat space with the metric form

$$dS_3{}^2 = dZ^2 - dX^2 - dY^2 \tag{11}$$

The embedding equations are

$$X = Rr/(1 - \tfrac{1}{4}tr^2) \tag{12}$$

$$Y = \tfrac{1}{2}CR^2r^2/(1 - \tfrac{1}{4}tr^2) \tag{13}$$

$$Z = \tfrac{1}{2}Rr^2(C^2R^2 + t)^{1/2}/(1 - \tfrac{1}{4}tr^2) \tag{14}$$

The equation of the (t, r) surface can now be written as

$$Z^2 - X^2 - (Y - 1/C)^2 = -1/C^2 \tag{15}$$

Thus the 2-space of interest is a subset of the one-sheet hyperboloid of revolution given by (15). The hyperboloid has the symmetry axis parallel to the Z axis and piercing the (X, Y) plane in the point $(X, Y) = (0, 1/C)$, in the pseudo-Euclidean space of metric (11). The parametrization (12)-(14) covers only that part of the hyperboloid (15) on which $\{Y > 0, Z > 0\}$ (where $|r| < 2/\sqrt{t}$ for $t > 0$) and $\{Y < 0, Z < 0\}$ (where $t > 0$ and $|r| > 2/\sqrt{t}$). $Z = 0$ corresponds to $t = -C^2R^2$, while $Y = 0$, where $r = 0$ or $t = \infty$, is a singular locus of the parametrization (12)-(14). The points where $t > 0$ and $|r| = 2/\sqrt{t}$ are at $\pm\infty$.

Let us now describe the foliation of the hyperboloid (see Figures 1 and 2). From (13) and (14) we see that

$$Z = [(C^2R^2 + t)^{1/2}/CR] \cdot Y \tag{16}$$

i.e., the $\{t = \text{const}\}$ lines are lines of intersection of the hyperoloid with the planes given by (16), all of which contain the X axis. The $t = -C^2R^2$ line is the $Z = 0$ cut, i.e., the equatorial circle of the hyperboloid. The other $(t = \text{const})$ lines with $t < 0$ are ellipses. The $t = 0$ line is the $Z = Y$ cut which is a parabola.

[3]The condition R = const is responsible for the singularity at $r = 0$. With R = const this singularity appears also in the general case, see (6). It is the special case of the singularity which appears whenever the function $(\partial/\partial t)\,(V/R)$ has a zero; see Sections 5 and 6.

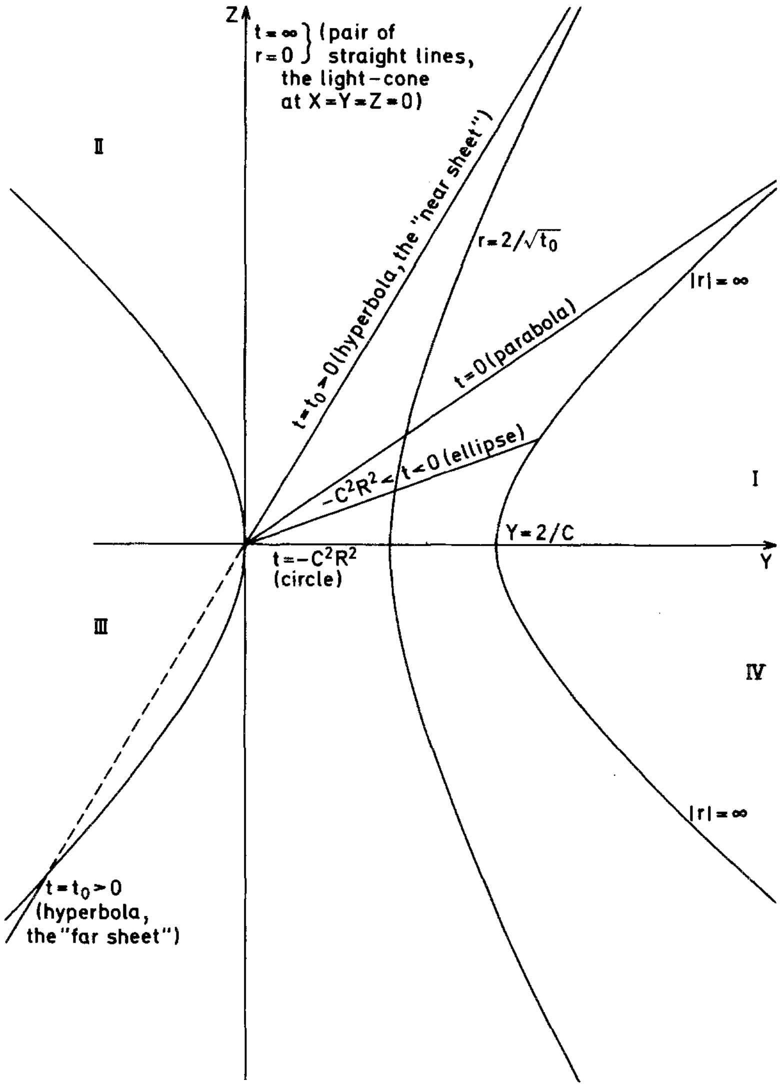

Fig. 1. Projection of the hyperboloid representing the de Sitter manifold onto the (Y, Z) plane. The straight lines are actually side views of the ellipses, the parabola and the hyperbolae, as indicated. The parts of the hyperboloid lying in the Sectors II and IV are not covered by the paremetrization of equations 12–14. Note that each section $t = t_0 > 0$ consists of two sheets. Each section r = const also consists of two sheets, the second of which (not shown) always passes through $X = Y = Z = 0$.

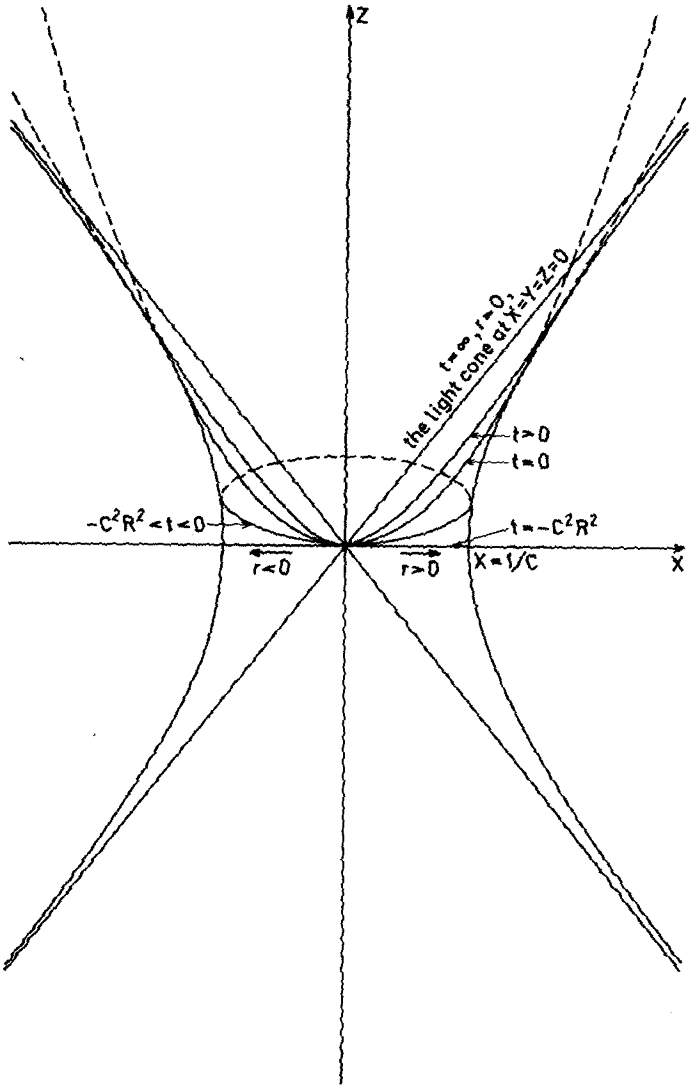

Fig. 2. *The same hyperboloid projected onto the (X, Z) plane.* Intersections of different t = const subspaces are artifacts *resulting from* suppressing one dimension. *The plane* $Y = 0$ which corresponds to *the singular limits* $r \to 0$ *and* $t \to \infty$ intersects the hyperboloid along the light cone at $X = Y = Z = 0$. This shows that all other t = const sections are indeed spacelike.

Other (t = const) lines with $t > 0$ are hyperbolas. In the limit $t \to \infty$, as already mentioned, the (t, r) parametrization becomes singular. The formulas (12)-(14) ascribe to $t = \infty$ the single point $X = Y = Z = 0$ while equations (15) and (16) would make us expect that $t \to \infty$ corresponds to $Y \to 0$, i.e., to the pair of straight lines $Z^2 - X^2 = 0$ which result from cutting the hyperboloid by the plane $Y = 0$. We will explain the reason of this singularity below.

From (12) and (13) we see now that

$$Y = \tfrac{1}{2} CRrX \tag{17}$$

and this shows that the r = const lines (i.e., the t lines which coincide with world lines of matter in the general Stephani solution) are lines of intersection of the hyperboloid with the planes (17), all of which contain the Z axis. The limit $r \to 0$ which corresponds to the plane $Y = 0$ is again singular, the limits $r \to \infty$ and $r \to -\infty$ both correspond to the same hyperbola $\{X = 0, C^2(Y - 1/C)^2 - C^2Z^2 = 1\}$.

The (t, r) parametrization of the de Sitter space-time used in (12)-(13) has several unusual features which we will later find in the general Stephani solution. We shall now briefly describe them.

We can go along each r-coordinate line from $r = 0$ to $|r| = \infty$ only if $t \leqslant 0$. If we take any $t = t_0 > 0$ section of the hyperboloid, then the hyperbola will escape to future infinity $\{Y \to \infty, Z \to \infty\}$ at the finite value of $|r| = 2/(t_0)^{1/2}$ without crossing any of those t lines for which $|r| \geqslant 2/(t_0)^{1/2}$. On those lines, the value of the t coordinate can never reach t_0. In other words, in this foliation each t line except $|r| = \infty$ will have a finite maximal value of t at future infinity. This maximum tends to zero as $r \to 0$, but at $r = 0$ the value of the t coordinate becomes indefinite since the $r = 0$ lines coincide with the $t = \infty$ lines, and so the $t = \infty$ hypersurface fails to intersect any of the t lines in any definite point. This is the meaning of the $t = \infty$ coordinate singularity. Incidentally, the singular lines $t = \infty$ $(r = 0)$ coincide with the light cone at $X = Y = Z = 0$. All the other sections t = const are thus indeed spacelike.

Note that if, at $t = t_0 > 0$, we let $|r|$ continue beyond $r_B(t_0) \stackrel{\text{def}}{=} 2/(t_0)^{1/2}$, then we jump from the future infinity on the hyperboloid, where $Z = +\infty$, to the past infinity where $Z = -\infty$. Thus in fact each t = const > 0 hypersurface consists of two sheets, one of which lies in the $\{Y \geqslant 0, Z \geqslant 0\}$ sector of the hyperboloid and contains the neighborhood of the point $r = 0$, the other lies in the sector $\{Y < 0, Z < 0\}$ and contains the neighborhood of the point $r = +\infty$ (which actually coincides with $r = -\infty$). We shall sometimes refer to them as the "near sheet" and the "far sheet," respectively. When $t \to \infty$ the sheets touch each other at $X = Y = Z = 0$ and form a pair of straight lines.

If we explore the Universe from within a certain $t = t_0 > 0$ space (of negative curvature) starting from $r = 0$, we find that some of the t-coordinate lines (matter flow lines in the general case) which still intersected the t = const spaces at

$t < 0$ (when the curvature was positive), slipped out of the space and there is no way to reach them, even if we proceed to an infinite distance from the starting point. Moreover, as k decreases (t increases) being negative, more and more t lines slip out of the space. Simultaneously, with $k < 0$, a second sheet of each t = const space appears which is separated from the first one by a singularity of the affine distance $s = \int_{r_1}^{r_2} ds \,|_{t,\theta,\phi=\text{const}}$ and into which more and more t lines enter. We will find precisely this occurring in the general case.

Let us also note that each r = const hypersurface consists of two sheets, one of which contains the point $X = Y = Z = 0$. With $t > 0$ ($k < 0$) and $|r| > 2/\sqrt{t}$, the hypersurfaces t = const and r = const intersect in the sector $\{Y < 0, Z < 0\}$, while with $t < 0$ they intersected in the sector $\{Y > 0, Z > 0\}$. This is a warning that the t lines which keep entering the far sheet are not the same ones that escaped from the near sheet. Moreover, if we assume the time arrow on the de Sitter manifold to point toward $Z = +\infty$, then the time ordering of the near sheets with respect to the t coordinate will agree with this global one, but the t ordering of the far sheets will be reversed.

We have chosen the coordinates so that $k(t) = -t$ in order to be able to describe and display the foliation of the space-time in a simpler way. However, an arbitrary $k(t)$ can be introduced simply by the coordinate transformation $t = -k(t')$ in (9), (10), (12)-(14), and (16). It is seen then that with arbitrary $k(t)$ the tilt of the sections t = const with respect to the XY plane need not change monotonously with t. In particular, with the function $k(t)$ having zeros at several values of t, the t = const sections of the hyperboloid corresponding to these values of t will all lie in the plane $Y = Z$, i.e., will coincide with the parabola representing the section of zero curvature. Such a foliation is rather ugly, since several values of the t coordinate will correspond to the same space-time point. This is why the choice $k(t) = -t$ is in fact the only reasonable one to describe the change from positive to negative spatial curvature [the opposite change would be described with $k(t) = +t$].

§(4): *The Geometry of the 3-Spaces t = const in the Vicinity of $t = t_{op}$*

We shall now show that the Stephani Universe shares many qualitative features with the de Sitter space-time foliated as in the previous section. For ease of discussion we will deal with another special case: the spherically symmetric space-time which results from (1)-(4) on setting x_0, y_0, z_0 const, and then transforming the constants away by a choice of coordinates. We shall then change to spherical coordinates in which

$$ds^2 = D^2\, dt^2 - \frac{R^2(t)}{V^2(t,r)}\,[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{18}$$

$$V = 1 + \tfrac{1}{4}\, k(t)\, r^2 \tag{19}$$

the equations (3) and (4) being still valid. The solution (18), (19) is a special case of those considered by Kustaanheimo and Qvist [15] (quotation after [16]).

Let us consider the family of the radial lines $(\theta, \phi) = \text{const}$ emanating from the center of symmetry $r = 0$ in an arbitrary spacelike section $t = t_0$. The length of the radial line from the point $r = 0$ to the point $r = r_0$ is equal to

$$l(t_0) = \int_0^{r_0} \frac{R(t_0)}{V(t_{0,r})} dr$$

$$= \begin{cases} 2R(t_0)[k(t_0)]^{-1/2} \arctan \{\frac{1}{2} [k(t_0)]^{1/2} r_0\} \stackrel{\text{def}}{=} l_c & \text{if } k(t_0) > 0 \quad (20) \\ r_0 R(t_0) \stackrel{\text{def}}{=} l_F \quad \text{if } k(t_0) = 0 & (21) \\ \dfrac{R(t_0)}{[|k(t_0)|]^{1/2}} \ln \dfrac{1 + \frac{1}{2} [|k(t_0)|]^{1/2} r_0}{1 - \frac{1}{2} [|k(t_0)|]^{1/2} r_0} \stackrel{\text{def}}{=} l_{\text{op}} & \text{if } k(t_0) < 0. \quad (22) \end{cases}$$

The solution (18), (19) is now seen to share the following properties with the de Sitter space-time of Section 3.

Theorem 4.1. Every section $t = t_0$ with $k(t_0) > 0$ is, as a three-dimensional Riemannian space, extendible to a 3-sphere, and so is of finite volume.

Proof. With $t = t_0$ and $k(t_0) > 0$ equation (18) represents a space in which the r lines are geodesics. The geodesic distance from $r = 0$ to $r = r_0$ within the space is a C^∞ function of r_0 [equation (20)] for $0 \leqslant r_0 \leqslant \infty$, and so the *space* can be extended to the whole range $0 \leqslant r \leqslant \infty$ even if there are *space-time* singularities at some values of r at $t = t_0$. The subspace $r \to \infty$ in the space-time is in general a single line on which t is the parameter; see (18), (19), and (3) (the line may degenerate to a point with unsuitable choice of k and R). In any case, the intersection of the subspace $r \to \infty$ with the subspace $t = t_0$ is a single point. Thus the completed space $t = t_0$ will have the topology of a 3-sphere. It has also the metric of a 3-sphere with r being the stereographic coordinate. ∎

Theorem 4.2. Every section $t = t_0$ with $k(t_0) = 0$ is extendible to the geodesically complete space R^3.

Proof. Every geodesic of the space $t = t_0$ which starts at $r = 0$ can be continued to infinite values of r. No identifications of points of the space at finite values of r are allowed since this would introduce discontinuities in pressure [see equation (6)]. But $r \to \infty$ is at an infinite geodesic distance from $r = 0$ [equation (20)] and the space is intrinsically flat. Thus the only allowable extension is R^3. ∎

Theorem 4.3. Every section $t = t_0$ with $k(t_0) < 0$ is extendible to a space of infinite volume, but even so extended it does not intersect some matter flow lines.

Proof. The geodesic distance l_{op} from $r = 0$ to $r = r_0$ within the space is a C^∞ function of r_0 only if $0 \leqslant r_0 < r_B(t_0) \overset{\text{def}}{=} 2/|k(t_0)|^{1/2}$. Again, no identifications of points are allowed, but the geodesic escapes to infinity already at $r_0 \to r_B$. Thus the matter flow lines given by $r \geqslant r_B(t_0)$ do not intersect that connected component of the completed $t = t_0$ space which contains $r = 0$. For the completion of the proof see next theorem.

Theorem 4.4. Every section $t = t_0$ with $k(t_0) < 0$ consists actually of two disjoint sheets, none of which contains the points $r = r_B(t_0)$.

Proof. This is simultaneously a continuation of the proof of Theorem 4.3. Although we run out to infinity ($l_{op} \to \infty$) by approaching $r = r_B(t_0)$ from below, the metric (18) and curvature determined by (5), (6) are well behaved also for $r > r_B(t_0)$ [except possibly for those points where $(\partial/\partial t)(V/R) = 0$—see Sections 5 and 6]. [Recall: (5), (6) determine the curvature since the Weyl tensor = 0.] The metric and curvature determine only V^2, not V itself. In (22) we have taken $(V^2)^{1/2} = 1 - \frac{1}{4}|k|r^2$, but for $r > r_B$ (where $V < 0$) the appropriate expression is $(V^2)^{1/2} = \frac{1}{4}|k|r^2 - 1$. The geodesic distance between the points (t_0, r_1, θ, ϕ) and (t_0, r_2, θ, ϕ) where $r_B(t_0) < r_1, r_2 < \infty$ is then

$$l_{12} = \frac{1}{2} r_B \left(\ln \frac{r_2 - r_B}{r_2 + r_B} - \ln \frac{r_1 - r_B}{r_1 + r_B} \right) \tag{23}$$

This is well defined for $r_2 \to \infty$, but $l_{12} \to \infty$ if $r_1 \to r_B$. Thus the sheet of the space $t = t_0$ which contains $r = \infty$ (let us call it the "far sheet" again) does not contain the points for which $r \to r_B$. Note that also in this case $r \to \infty$ is in general a single line in the space-time and always a single point in the space $t = t_0$. ■

For future reference it should be noted that the values $r = r_B(t)$ are zeros of the function V.

Note that the far sheet is, as a three-dimensional Riemannian space, isometric to the near sheet. The isometry transformation is $r = 4/kr'$ with r' as the new r coordinate. In the FRW models, this is an isometry of the space-time itself, and so by performing it we simply obtain another copy of the same space-time. Not so here: as seen from (1) and (6) the metric component g_{00} and the pressure are not invariant under this transformation.

§(5): *The Foliation of the Space-Time*

We will still consider the spherically symmetric subcase. The lapse of the proper time between the hypersurfaces $t = t_1$ and $t = t_2 > t_1$ along the line $(r, \theta, \phi) = (r_0, \theta_0, \phi_0)$ is given by

$$s_{12}(r_0) = -\int_{t_1}^{t_2} D(t, r_0)\, dt = -\int_{t_1}^{t_2} F(t) \left[\ln\left(\frac{V}{R}\right) \right]_{,t} dt \tag{24}$$

If the function $I(t,r) \stackrel{\text{def}}{=} [\ln(V/R)]_{,t}$ has a constant sign along the t line in the interval $t_1 \leqslant t \leqslant t_2$, and $F(t)$ is continuous in that interval, then by applying the mean-value theorem to (24) we obtain

$$s_{12}(r) = -F(t_m(r)) \ln \frac{|V(t_2,r)|R(t_1)}{|V(t_1,r)|R(t_2)} \tag{25}$$

where $t_1 \leqslant t_m(r) \leqslant t_2$. This equation allows to establish a few more properties of the Stephani space-time which it shares with the de Sitter solution of Section 3.

Theorem 5.1. Let $k(t_2) < 0$, $k(t_1) > k(t_2)$, $r_1 < r_B(t_2) = 2/[-k(t_2)]^{1/2}$, and let $|t_2 - t_1|$ and $|r_1 - r_B(t_2)|$ be sufficiently small so that $I(t,r)$ has a constant sign for $t_1 \leqslant t \leqslant t_2$ and $r_1 \leqslant r \leqslant r_B(t_2)$. Then $s_{12}(r) \to \infty$ as $r \to r_B(t_2)$, i.e., the infinity of the near sheet of the space $t = t_2$ is also infinitely far into the future from any other space $t = \text{const}$ along the flow line $r = r_B(t_2)$.

Proof. Under the assumptions stated $0 < V(t_1,r) < \infty$ for $r_1 \leqslant r \leqslant r_B(t_2)$, but $V(t_2,r) \to 0$ as $r \to r_B(t_2)$. Thus $s_{12}(r) \to \infty$ there. ■

Theorem 5.2. Let $k(t_2) = 0$, $k(t_1) > 0$, and let $|t_2 - t_1|$ be sufficiently small so that such r_L exists that $I(t,r)$ has a constant sign for $r_L < r < \infty$ and $t_1 \leqslant t \leqslant t_2$. Then $s_{12}(r) \to \infty$ as $r \to \infty$, i.e., the infinity of the flat space $t = t_2$ is infinitely far to the future from any space $t = t_1 < t_2$ along the matter flow line $r \to \infty$.

Proof. We have $V(t_2,r) \equiv 1$ while $V(t_1,r) \geqslant 1$ for all r, and $V(t_1,r) \to \infty$ as $r \to \infty$. ■

Theorem 5.3. Let $k(t)$ be continuous and bounded in the range $t_I \leqslant t \leqslant t_F$, become negative somewhere in that range, take its local minimum in $[t_I, t_F]$ at $t = t_M$, and its local maximum at $t = T_M$, $t_I \leqslant T_M < t_M \leqslant t_F$. Then there exist such matter flow lines $r = r_0$ on which a finite value of the t coordinate corresponds to future timelike infinity, $s_{12}(r_0) \to \infty$.

Proof. Obviously $k(t_M) < 0$. Let $2/|k(t_M)|^{1/2} < r_0$ and simultaneously $r_0 < 2/|k(T_M)|^{1/2}$ if $k(T_M) < 0$ or $r_0 < \infty$ if $k(T_M) \geqslant 0$. Then, from continuity and boundedness of $k(t)$ it follows that such t_0 exists at which $k(t_0) < 0$ and $r_0 = 2/|k(t_0)|^{1/2}$, $T_M \leqslant t_0 \leqslant t_M$. Let us choose t_1 close enough to t_0 so that $I(t,r_0)$ has a constant sign for $t_1 \leqslant t \leqslant t_0$, and (25) applies. Let us calculate (25) along the flow line $r = r_0$, and let us increase t_2 starting from $t_2 = t_1$ if $t_1 \geqslant t_I$ or from $t_2 = t_I$ otherwise. Then $s_{12}(r_0) \to +\infty$ as $t_2 \to t_0$. ■

Theorem 5.4. Let the notations of Theorem 5.3 hold, but let $t_I \leqslant t_M < T_M \leqslant t_F$. Then along some flow lines $r = r_0$ a finite value of t corresponds to past timelike infinity, $s_{12}(r_0) \to -\infty$.

Proof. The following adaptations in the proof of Theorem 5.3 must be made:

(1) This time $t_M \leqslant t_0 \leqslant T_M$.
(2) Choose t_2 close enough to t_0 so that $I(t, r_0)$ has a constant sign for $t_0 \leqslant t \leqslant t_2$.
(3) Let us decrease t_1 starting from $t_1 = t_2$ if $t_2 \leqslant t_F$ or else from $t_2 = t_F$.
(4) $s_{12}(r_0) \to -\infty$ as $t_1 \to t_0$. ∎

Let us recall that in the de Sitter space-time, represented as in Section 3, the slices t = const all met in the point $r = 0$ which was a singularity of the foliation. Can different t = const spaces meet (i.e., intersect or be mutually tangent) in the general case?

The hypersurfaces $t = t_1$ and $t = t_2$ meet at $r = r_C$ if $s_{12}(r_C) = 0$. Unfortunately, the mean-value theorem becomes inapplicable precisely in this case, and so there is no way to calculate the zeros of $s_{12}(r)$ in general. However, we can investigate when two infinitesimally close hypersurfaces t = const meet. This happens when $ds = 0$ along a matter world line (which is known to be timelike) at $r = r_C$. We can see from (1) and (3) that this happens at the zeros of the function $(\partial/\partial t)(V/R)$. At these zeros, as seen from (6), the pressure becomes infinite, and so a curvature singularity occurs. Thus two spacelike slices having a common point is a truly singular event.

There are two ways to avoid such a singularity: (1) To let the zeros of $(\partial/\partial t)(V/R)$ coincide with the zeros of V, so that the factors responsible for the singularity cancel out in (6). (2) Not to let $(\partial/\partial t)(V/R)$ have zeros at all. In the first case a FRW model is obtained, so this is not an interesting way out of the trouble. The second case will be discussed in Section 6.

§(6): *Properties of Matter*

In this section we discuss the general case (with no symmetry) at the beginning.

The moments when the Universe opens or closes are characterized by

$$k(t_{\mathrm{op}}) = k(t_c) = 0. \tag{26}$$

Some scalars of the matter flow do not "feel" these moments as any special events, e.g., $\theta = u^{\alpha}{}_{;\alpha} = -3/F(t)$ or $\kappa\epsilon = 3C^2(t)$ can behave there as regularly as one likes. The acceleration vector field has its scalar equal to

$$\begin{aligned} a^2 = -g_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta} = &\left[2R^2 \frac{\partial}{\partial t}\left(\frac{V}{R}\right)\right]^{-2} \sum_{j=1}^{3} \left[V^2 k^2 \dot{x}_{0j}^2 - 2k\dot{k}\dot{x}_{0j}(x_j - x_{0j})\, V \right. \\ &\left. + \dot{k}^2 (x_j - x_{0j})^2 + k^2 \dot{k}(x_j - x_{0j})^2 \sum_{l=1}^{3} (x_l - x_{0l})\,\dot{x}_{0l}\right] \\ &- k\left[k/2R^2 \frac{\partial}{\partial t}\left(\frac{V}{R}\right)\right]^2 \cdot \left[\sum_{j=1}^{3} (x_j - x_{0j})\,\dot{x}_{0j}\right]^2 \end{aligned} \tag{27}$$

From this it can be seen that a^2 becomes infinite at the zeros of $(\partial/\partial t)\,(V/R)$, i.e., precisely where pressure has its singularity [see (6)], but is finite everywhere else, including $x_i \to \infty$.

In the spherically symmetric case, where x_{0j} are constant, the matter flow line $|x_i| = \infty$ is a geodesic, just as the line $x_i = x_{0i}$.

Let us denote the zeros of the function V by $x_i = N_i(t)$. At $x_i = N_i$ we have $\kappa p = -3C^2 = -\kappa\epsilon < 0$. This happens with $k < 0$ (with $k > 0$, V has no zeros; we will discuss that case further). The negative values of p can then be avoided in two ways:

1. Letting $(\partial/\partial t)\,(V/R)$ have its zeros where V does. This occurs only with $k = \text{const}$, i.e., only in the classical FRW models.

2. Matching the Stephani solution to an empty-space solution so that the matter-filled region does not reach up to $x_i = N_i(t)$ in any hypersurface. This matching will not be discussed in this paper. Let us only observe that the exterior solution cannot belong to the family (1)-(6), just because (1)-(6) is conformally flat: each Ricci-flat solution of that family would be the Minkowski space which cannot be reasonably expected to be the exterior metric. However, one necessary condition for the existence of an exterior solution is fulfilled here: on the boundary of the Stephani space-time one should have $p = 0$ along a famiy of matter lines, e.g., $p = 0$ for $x_i = x_{si}$. For any constant value of x_{si} the equation $p(t, x_{si}) = 0$ is just a single ordinary differential equation connecting the functions C, k, R, x_{0i} as seen from (6). Thus $p = 0$ can be guaranteed at any constant values of x_i for all t.

The conditions for p to be positive and finite everywhere we shall again discuss in the spherically symmetric subcase.

When $k(t_0) > 0$, the pressure can behave regularly throughout the hypersurface $t = t_0$, as can be seen from (6). At $r = 0$ it equals

$$\kappa p(t,0) = -3C^2 - 2\dot{C}R/\dot{R}, \tag{28}$$

and is nonnegative if

$$-\tfrac{2}{3}\,\dot{C}R/C\dot{R} \geqslant 1 \tag{29}$$

If the function $(\partial/\partial t)\,(V/R)$ has no zeros, then pressure remains finite for all r. That happens if

$$\dot{R}(R\dot{k} - \dot{R}k) < 0. \tag{30}$$

The condition (29) must be fulfilled also in a FRW model. With $k > 0$, (30) is fulfilled when $k = \text{const}$, and so will be fulfilled if the derivative of k is sufficiently small for all t.

With (29) and (30) fulfilled, the condition $p \geqslant 0$ is equivalent to

$$L \overset{\text{def}}{=} -\tfrac{3}{4}\,C^2\dot{R}(\dot{k}R - k\dot{R}) + \tfrac{1}{2}\,kC\dot{C}R\dot{R} \leqslant 0 \tag{31}$$

Equation (29) implies that

$$L \leqslant -\tfrac{3}{4}\,C^2\dot{k}R\dot{R} \tag{32}$$

Thus the sufficient (but not necessary) condition for (31) to hold is

$$\dot{k}R\dot{R} \geqslant 0. \tag{33}$$

It is now clear that with $k < 0$, (30) and (33) cannot hold simultaneously, i.e., p is not guaranteed to be nonengative everywhere. With $k > 0$ it can only be said that (30) and (33) are not contradictory, and are automatically fulfilled when k = const. It is, however, a separate question whether (29), (30), and (33) can hold simultaneously over long ranges of t. This question will be addressed in a separate paper.

§(7): *Conclusions*

From the foregoing considerations we can conclude the following:

1. In the spherically symmetric case [and also in the general case, if the functions $x_0(t), y_0(t)$, and $z_0(t)$ are sufficiently small everywhere] there will exist a world tube around the central line $r = 0$ (resp. $x = y = z = 0$) in which none of the exotic effects (like matter flow lines slipping out of the space or pressure becoming negative or unbounded) will occur. The region outside the tube can be removed from the space-time by matching the Stephani solution to an empty space solution which still remains to be found (in the spherically symmetric subcase it is just the Schwarzschild solution).

2. One can also consider such Stephani models in which the function $k(t)$ has a fixed sign forever. If, moreover, $k > 0$, then each space t = const will intersect all matter flow lines, and the evolution of such an Universe will be qualitatively very similar to the evolution of the closed FRW Universe, but nevertheless the model will have a more complicated geometry which still deserves consideration.

To decide whether the Stephani solution qualifies as a model of the observed Universe is a task for further works. It must be stressed however that the classical FRW models are contained in it as special cases. Thus by choosing the arbitrary functions appropriately one can make the predictions of the Stephani model arbitrarily close to the predictions of the FRW models. Consequently, no astronomical observations made so far can rule out the possibility that our physical Universe is of such a more general type: their continuing low precision and dependence on *a priori* assumptions leave enough room for this generalization. The reason for which only the FRW solutions were used as cosmological models up to now is in the theory, not in its observational basis. The Stephani model was ruled out at the very start when the "cosmological principle," i.e., homogeneity of the space-time, was taken as an axiom. Indeed, as indicated by Ellis [17], this is a philosophical principle with an insufficient observational foundation. To rule out, or admit, the Stephani solution as a cosmological model one has to analyze the qualitative differences between the Stephani model and the FRW models, and decide whether they are in agreement with observations. For instance, one might try to look for physical consequences of the change of sign of curvature, or of the singularity which appears with negative curvature. This remains to be done.

Acknowledgments

The author is grateful for enlightening discussions and comments to J. Ehlers, B. G. Schmidt, G. F. R. Ellis, M. Walker, and J. Bicak, and also to R. Breuer and M. Walker for their help in locating the references. The cooperation of the authors of Refs. 6, 7, 8, 10, 12, and 13 in preparing Section 1 is appreciated.

The author wishes to thank the Max-Planck-Institute for Physics and Astrophysics in Garching for providing perfect working conditions, and the Alexander von Humboldt Foundation for its generous support during the period of fellowship.

References

1. Friedman, A. A. (1922). *Z. Phys.*, **10**, 377; (1924). *Z. Phys.*, **21**, 326.
2. Robertson, H. P. (1929). *Proc. Natl. Acad. Sci.*, **15**, 822; (1933). *Rev. Mod. Phys.*, **5**, 62; Walker, A. G. (1935). *Q. J. Math. Oxford*, Ser. **6**, 81.
3. Stephani, H. (1967). *Commun. Math. Phys.*, **4**, 137.
4. Krasiński, A. (1981). *Gen. Rel. Grav.*, **13**, 1021.
5. Collins, C. B. (1979). *Gen. Rel. Grav.*, **10**, 925.
6. Mashhoon, B., and Partovi, M. H. (1980). *Ann. Phys. (N.Y.)*, **130**, 99.
7. Bergmann, O. (1982). *Phys. Lett. A*, **82**, 383.
8. Schmidt, H. J. (1982). *Astron. Nachr.*, **303**, 227.
9. Geroch, R. (1967). *J. Math. Phys.*, **8**, 782.
10. Kundt, W. (1967). *Commun. Math. Phys.*, **4**, 143.
11. Brill, D. R. (1972). In *Magic without magic: John Archibald Wheeler*, Klauder, J. R., ed. (W. H. Freeman and Co., San Francisco), p. 309.
12. Yodzis, P. (1972). *Commun. Math. Phys.*, **26**, 39.
13. Yodzis, P. (1973). *Gen. Rel. Grav.*, **4**, 299.
14. Schrödinger, E. (1956). *Expanding Universes* (Cambridge University Press, Cambridge), pp. 14 and 15.
15. Kustaanheimo, P., and Qvist, B. (1948). *Societas Scientarium Fennica, Commentations Physico-Mathematicae (Helsingfors)* **13**, 16.
16. Kramer, D., Stephani, H., Herlt, E., and MacCallum, M. (1980). *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge), p. 168.
17. Ellis, G. F. R. (1979). *Gen. Rel. Grav.*, **11**, 281.