THE UNIVERSE WITH VARYING TOPOLOGY OF SPATIAL SLICES(*)

Andrzej Krasiński

Department of Physics, Konstanz University, 7750 Konstanz, W. Germany

(On leave of absence from N. Copernicus Astronomical Center, Warsaw, Poland)

1. Introduction.

The Friedman-Robertson-Walker (FRW) cosmological models are derived from general relativity under the assumption that the spacetime is homogeneous and isotropic. While the geometrical meaning of this assumption is clear, its justifications are less satisfactory. They range from overstretching the Copernican principle (no region of the Universe should be geometrically distinguished, "therefore" all regions of equal size must be identical) to the confession that the FRW models are the only exact solutions in Einstein's theory which allow to exactly calculate various cosmological predictions of the theory. The observations do not always seem to certify the homogeneity of the actual Universe. The discovery of voids and clusters in the distribution of galaxies [1] shows that the Universe may possibly be homogeneous only on such large scales on which the observations are not sufficiently precise to be 100% reliable. On the other hand, all models of galaxy formation are based on perturbations of the FRW models, since the FRW solutions taken literally are telling us that no galaxies may have formed.

All this cries for generalizations: less regular models which would contain the FRW ones as special cases, but be capable of describing more phenomena. This note presents a way of searching for such generalized models. It should be observed that this author is not the first one to feel uneasy about the way the FRW models are tested [2].

2. The Stephani model.

It is enough to weaken the assumptions of homogeneity and isotropy only slightly to obtain the first interesting generalization. Let us assume that only each section t = const of the spacetime is homogeneous and isotropic, but the symmetry groups of these 3-spaces are not necessarily symmetry groups of the whole spacetime (this is an example of Collins's intrinsic symmetries [3]). Let us assume also that matter moves along the t-lines which are orthogonal to the 3-spaces t = const and is a perfect fluid - as it was in the FRW models. Under these assumptions the Einstein field equations yield the following solution:

$$ds^2 = D^2 dt^2 - (R^2/V^2)\,(dx^2 + dy^2 + dz^2), \quad \text{where} \tag{1}$$

$$D = F\cdot(\dot{R}/R - \dot{V}/V), \tag{2}$$

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$$V = 1 + \frac{1}{4} k\, (x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 \,, \tag{3}$$

$$k = (C^2 - 1/F^2)R^2 \,, \tag{4}$$

C, F, R, x_o, y_o, z_o being arbitrary functions of t. The energy density is

$$\kappa\varepsilon = 3C^2 , \tag{5}$$

$\kappa = 8\pi G/c^2$, and the pressure is

$$\kappa p = - 3C^2 + 2C\dot{C}V/R \frac{\partial}{\partial t} (V/R) . \tag{6}$$

This solution was first obtained by Stephani [4], then reobtained and investigated by this author [5]. It reduces to a FRW model under any one of the following conditions: (a) k, x_o, y_o, z_o are all constant; (b) Pressure depends only on time; (c) An equation of state of the form $\varepsilon = \varepsilon(p)$ holds; (d) Matter flow-lines are geodesics. In general, the solution is different from the FRW models as certified by the following invariant properties: I. With arbitrary x_o, y_o, z_o the spacetime has no symmetry; II. The pressure and the equation of state are position-dependent; III. Matter-content of the Universe moves with acceleration.

3. Properties of the Stephani model.

A characteristic property of this model is the fact that the curvature index k is a function of time which can change its sign. It means, some spaces t = const have positive constant curvature and so are closed, while some others have negative constant curvature and are open (these properties are proven and explained in [6]). There exists a simple example of such geometry: the de Sitter spacetime which results from (1)-(6) when C = const, i.e. $3C^2 = \Lambda$. This spacetime is represented in a reduced embedding diagram as a hyperboloid. Its foliation into spaces t = const induced by (1)-(6) consists of intersections of this hyperboloid with planes tilted at different angles to its equator. The tilt is determined by the value of the function $k(t)$. For $k(t) > 0$ the lines of intersection are ellipses, for $k(t) < 0$ they are hyperbolae, for $k(t) = 0$ it is a parabola (see [6] for a detailed discussion with figures). The general Stephani model is shown in [6] to share several qualitative properties with the de Sitter spacetime so foliated.

4. The need of further generalizations.

Although the Stephani model gives us an insight into the mathematically possible generalizations of the FRW models, it is not exactly the generalization we argued for in the introduction. The energy-density (5) is spatially uniform, so the formation of galaxies still cannot be described in a non-perturbative manner. However, the pressure (6) is already position-dependent what is a step in the right direction. It might thus be sensible to try to generalize the Stephani solution further. The following line of reasoning seems promising. Let us assume that the spacetime can be foliated into conformally flat sections t = const which are orthogonal to the t-lines. Its metric form is then again given by (1) where D and V are now arbitrary functions and $R = R(t)$. Let us assume also that the source in the field equations is a perfect fluid which moves along the t-lines. Then the Einstein field equations imply eq. (2) (from the orthogonality of the velocity field to the t = const sections), and in addition

$$(R/V^2)\, V_{,ij} = F_k(x,y,z) \qquad (7)$$

from diagonality of the energy-momentum tensor, and

$$(R/V^2)\, (V_{,ii} - V_{,jj}) = G_k(x,y,z) \qquad \text{(no summation)} \qquad (8)$$

from the equality of three eigenvalues of the energy-momentum tensor, where $i, j, k = 1, 2, 3$ run cyclic, $x^1 = x$, $x^2 = y$, $x^3 = z$, and $F_1, \ldots, G_3$ are arbitrary functions of the spatial coordinates. The Stephani model follows as the special case $F_k = G_k = 0$. Let us note now that each equation $G_k = 0$ is a "wave" equation in two spatial coordinates which thus admits periodic functions as solutions. Since ε is a function of V and its derivatives, each such solution would imply a periodic spatial variation of the matter-density. This is a very attractive feature. Unfortunately, no explicit solution was found so far, and it is not even certain whether $G_k = 0$ may co-exist with $F_k \neq 0$. However, the author considers this line of investigation being worth a pursuit. It must be stressed that the Stephani solution and its generalizations do not necessarily imply the change of sign of spatial curvature. The function k in (4) may well have a constant sign; what is essential to prevent (1)-(6) from reducing to a FRW model are only the acceleration field and spatial gradients of pressure.

References

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