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SYMMETRIES OF THE RIEMANN TENSOR(*)

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1. Symmetries of a peculiar Newtonian field.

There exists a Newtonian gravitational field [1] which has a kind of inner symmetry. Let us display it in the coordinates (r, θ, ϕ) given by

$$x = (r^2 + a^2)^{1/2} \sin\theta \cos\phi, \quad y = (r^2 + a^2)^{1/2} \sin\theta \sin\phi, \quad z = r \cos\theta, \tag{1}$$

where x, y, z are cartesian coordinates and a = const. The r = const surfaces are confocal ellipsoids of revolution with common eccentricity a,

$$V_e(r) = -\frac{GM}{a} \arctan \frac{a}{r} \tag{2}$$

is the potential outside the source, where $M = 4\pi \int_0^{r_o} f(r) dr$ is the mass of the source, $f(r)$ is an arbitrary function, $\rho(r,\theta) = f(r)/(r^2 + a^2 \cos^2\theta)$ is the density distribution in the source with outer surface at $r = r_o$, and

$$V_i(r) = 4\pi G \int_0^{r_o} \frac{dr'}{r'^2 + a^2} \int_0^{r_o} f(r'') dr'' \tag{3}$$

is the potential inside the source (for details see [1]). Let us consider the following motions of the space: 1. Each point retains its previous value of the r-coordinate, but is moved parallel to the (y,z) plane so that (a) Points with $x = 0$ move by $\Delta\theta$ counterclockwise (i.e. towards smaller θ if $\phi = 0$, towards bigger θ if $\phi = \pi$), (b) Points with $x \neq 0$ (which move on ellipses similar to the $\{x = 0, r = \text{const}\}$ ellipse) are displaced between points corresponding to those in (a) under the similarity transformation; 2. An analogous motion parallel to the (x, z) plane. These motions reduce to rotations around the x and y axes, respectively, when $a = 0$. They are obviously not symmetry transformations of the space. However, their generators in the (r,θ,ϕ) coordinates, given by

$$J_{yz} = \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}, \quad J_{xz} = \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}, \tag{4}$$

leave the potentials (2) and (3) invariant. Moreover, they are formally identical to generators of rotations expressed in spherical coordinates. This is a hint that similar things may happen in general relativity where the geometrical meaning of coordinates is never specified in advance.

2. Why the Riemann tensor?

The gravitational field manifests itself in the relativity theory through the curvature tensor. It is thus logical to look for analogies to the situa-

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tion described above by investigating the cases where the Riemann tensor is invariant under a larger group than the corresponding metric tensor. Such a project was proposed long ago by Katzin, Levine and Davis [2], and followed by McIntosh et.al. in a series of papers [3-5]. Unfortunately, in most cases these curvature collineations appeared to be identical with special conformal motions where $\underset{k}{\pounds} g_{\alpha\beta} = \phi g_{\alpha\beta}$ with $\phi_{;\mu\nu} = o$ [5].

However, the form of the equation $\underset{k}{\pounds}$(Riemann tensor)= o depends on the positions of indices. In [2-5] only $\underset{k}{\pounds} R^{\alpha}{}_{\beta\gamma\delta} = o$ was investigated. Let us take $\underset{k}{\pounds} R^{\alpha\beta}{}_{\gamma\delta} = o$ instead.

3. Symmetries of $R^{\alpha\beta}{}_{\gamma\delta}$.

Let $R^{\alpha\beta}{}_{\gamma\delta}$ be any tensor having all the indicial symmetries of the Riemann tensor ($R^{\alpha\beta}{}_{\gamma\delta} = -R^{\beta\alpha}{}_{\gamma\delta}$, and so on). Then:

$\underset{k}{\pounds} R^{\alpha\beta}{}_{\gamma\delta} = 0$ for any arbitrary vector field k if and only if $R^{\alpha\beta}{}_{\gamma\delta} = R(\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta} - \delta^{\alpha}{}_{\delta}\delta^{\beta}{}_{\gamma})$ with R = const, i.e. if and only if $R^{\alpha\beta}{}_{\gamma\delta}$ represents a space of constant curvature. (5)

The group of curvature collineations of $R^{\alpha}{}_{\beta\gamma\delta}$ is in this case a subgroup of conformal motions. This example shows that the collineations of $R^{\alpha\beta}{}_{\gamma\delta}$ may be much more numerous than those of $R^{\alpha}{}_{\beta\gamma\delta}$.

A modification of (5) holds when a Riemannian space V_N contains subspaces of constant curvature S_n. Suppose V_N can be orthogonally decomposed into a congruence of S_n whose symmetry group is also a symmetry (sub-)group of the whole V_N. Let k be a tangent vector field on S_n. Then:

Any k generates a curvature collineation of $R^{\alpha\beta}{}_{\gamma\delta}$ on V_N. (6)

In particular, the collineations of $R^{\alpha\beta}{}_{\gamma\delta}$ in a spherically symmetric spacetime (congruence of S_2) must contain the subgroup generated by

$$k = F(\theta,\phi)\frac{\partial}{\partial\theta} + G(\theta,\phi)\frac{\partial}{\partial\phi}\ , \tag{7}$$

where (θ,ϕ) are spherical coordinates and (F,G) are arbitrary functions.

4. Some examples.

Here are the generators of $\underset{k}{\pounds} R^{\alpha\beta}{}_{\gamma\delta} = o$ for four simple solutions. Unexplained symbols denote arbitrary functions.

a) The Schwarzschild solution (in standard spherical coordinates):

$$k = F(t,r)\frac{\partial}{\partial t} + G(\theta,\phi)\frac{\partial}{\partial\theta} + H(\theta,\phi)\frac{\partial}{\partial\phi}\ . \tag{8}$$

b) The Lanczos solution [6]:

$$ds^2 = dt^2 + 2Cr\,dt\,d\phi + (C^2r^2 - \alpha)d\phi^2 - (1/4e^r)dr^2 - e^{-r}\,dz^2,$$

$$k = (At + D)\frac{\partial}{\partial t} + (A\phi + B)\frac{\partial}{\partial\phi} + K(z)\frac{\partial}{\partial z}\ , \tag{9}$$

where A, B, C, D, Λ = const, $\alpha = C^2r + \Lambda + \Lambda e^{-r}$. The symmetry group of this metric consists of translations in the t, ϕ and z directions.

c) The Nariai solution [7-8]:

$$ds^2 = p^2 dt^2 - L^2 (dr^2/r^2 + d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$k = F(t,r)\frac{\partial}{\partial t} + G(t,r)\frac{\partial}{\partial r} + H(\theta,\phi)\frac{\partial}{\partial\theta} + K(\theta,\phi)\frac{\partial}{\partial\phi}, \qquad (10)$$

where L = const, $p = a(t)\cos(\log\frac{r}{L}) + b(t)\sin(\log\frac{r}{L})$. Note that this solution is a cartesian product of the (t,r) and (θ,ϕ) spaces, both of which are of constant curvature, and see how it fits into theorem (6).

d) The Robertson-Walker metrics (with x, y, z as coordinates in S_3):

$$k = F(x,y,z)\frac{\partial}{\partial x} + G(x,y,z)\frac{\partial}{\partial y} + H(x,y,z)\frac{\partial}{\partial z}. \qquad (11)$$

5. The "spherically symmetric $R^{\alpha\beta}{}_{\gamma\delta}$".

In view of what was said before it would be interesting to know what metrics can generate such $R^{\alpha\beta}{}_{\gamma\delta}$ which are invariant with respect to the generators (4). The equations $\underset{k}{\mathcal{L}} R^{\alpha\beta}{}_{\gamma\delta} = 0$ yield then the solutions:

$R^{01}{}_{01}$, $R^{03}{}_{03} = R^{02}{}_{02}$, $R^{03}{}_{13} = R^{02}{}_{12}$, $R^{13}{}_{03} = R^{12}{}_{02}$, $R^{13}{}_{13} = R^{12}{}_{12}$, $R^{23}{}_{23}$ are all arbitrary functions of r.

$R^{01}{}_{23}$, $R^{02}{}_{03}$, $R^{02}{}_{13}$, $R^{12}{}_{03}$, $R^{12}{}_{13}$ are all of the form (function of r) $\bullet \sin\theta$

$R^{03}{}_{02} = -R^{02}{}_{03}/\sin^2\theta$, $R^{03}{}_{12} = -R^{02}{}_{13}/\sin^2\theta$, $R^{13}{}_{02} = -R^{12}{}_{03}/\sin^2\theta$,

$R^{13}{}_{12} = -R^{12}{}_{13}/\sin^2\theta$, $R^{23}{}_{01} = f(r)$, where $f(r)$ arbitrary.

The corresponding metrics must fulfil $g_{\rho[\beta}R^{\alpha\rho}{}_{\gamma\delta]} = 0$. This is a formidable set of algebraic equations which has 26 sets of $\{g_{\alpha\beta}, R^{\alpha\beta}{}_{\gamma\delta}\}$ as solutions other than the trivial ones (singular $g_{\alpha\beta}$ or a space of constant curvature). The solutions are too large to be displayed here. From among them such metrics must be chosen which generate the corresponding $R^{\alpha\beta}{}_{\gamma\delta}$ through the Christoffel symbols. The work is not yet finished.

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