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SPACETIMES WITH SPHERICALLY SYMMETRIC HYPERSURFACES

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Abstract

When discussing spherically symmetric gravitational fields one usually assumes that the whole spacetime is invariant under the O(3)-group of transformations. In this paper, the Einstein field equations are investigated under the weaker assumption that only the 3-spaces t=const are O(3)symmetric. The following further assumptions are made: 1. The t-lines are orthogonal to the spaces t=const, 2. The source in the field equations is a perfect fluid, or dust, or the Λ-term, or the empty space. 3. With respect to the center of symmetry the fluid source may move only radially if at all. Under these assumptions one solution with a perfect fluid source, found previously by Stephani, is recovered and interpreted geometrically, and it is shown that it is the sole solution which is not spherically symmetric in the traditional sense. The paper ends with a general discussion of cosmological models whose 3-spaces t=const are the same as in the Robertson-Walker models. No new solutions were explicitly found, but it is shown that such models exist in which the sign of curvature is not fixed in time.

1. Introduction

When discussing spherically symmetric gravitational fields in the theory of general relativity one usually assumes at the very outset that the whole spacetime is spherically symmetric, i.e. that the metric is invariant under the $O(3)$ group of transformations of coordinates. This assumption seems more restrictive than necessary, however, because what one has in mind while doing observations is only the geometry of the instantaneous 3-spaces $t=const$ whatever the choice of time is. The geometry of the spacetime can then be recognized only by indirect investigations, and it might be interesting to see what conclusions can be drawn if we assume that only each of the 3-spaces $t=const$ is $O(3)$- symmetric while for the whole spacetime it is not necessarily so.

This problem has a simple analogy in the geometry of 2-dimensional surfaces, where the $O(3)$ transformations have their analogs in the rotations around a fixed axis. Suppose you take a family of circles, which are all axially symmetric curves, and you stack them to form a 2-dimensional surface. The question is: is the surface obtained in this way necessarily axially symmetric? One thinks first of a cylinder, a cone, a hyperboloid, a plane, a spheroid or sphere, all of which are axially symmetric. The less trivial case is a torus. It can be obtained by stringing identical circles onto another circle, orthogonal to their planes. Although the torus is axially symmetric, its axis of symmetry does not coincide with the axes of symmetry of the small circles forming it. Indeed, it is now easy to imagine the torus being bent, so that the line joining the centers of the small circles becomes wavy, still being orthogonal to each circle's plane in the point where they cross. Clearly, such a wavy torus has no symmetry, though it is still formed by stringing the circles onto a line. Even more examples of nonsymmetric surfaces can be imagined if one allows that the circles are stringed onto a line which is not orthogonal to their planes at the point of crossing. For example, identical cricles stringed onto a straight line which is inclined to their planes will form an elliptical cylinder which has quite a different symmetry.

In this paper I want to imagine a spacetime as made of $O(3)$-symmetric 3-dimensional spacelike hypersurfaces stringed onto a timelike line orthogonal to them all, and to investigate the properties of such a spacetime if the Einstein field equations are fulfilled. We shall also assume that the velocity field of matter, u^{α} (if any matter is present), has only the u^t and u^r components, i.e. moves only radially if at all with respect to the line joining the centers of symmetry of the 3-spaces. This assumption is justified by the fact that transversal motions of matter could be easily revealed by any observer, and would thus constitute a too-obvious evidence for the lack of ordinary spherical symmetry in the spacetime. To the author, it seems more interesting to investigate such spacetimes with spherically symmetric hypersurfaces which are not too easily distinguished from ordinary spherically symmetric spacetimes, and thus might potentially serve as generalized cosmological models. However, one could go on without the assumption of purely radial motions and see what follows. This problem still awaits investigation.

The most general source in the field equations considered here will be a perfect fluid, whose special cases (in the mathematical sense) are: dust-like matter (pressure = 0), the Λ-term ("pressure" = const $\stackrel{\text{def}}{=} \Lambda \neq 0$, "energy density" = $-\Lambda$), and the pure empty space (pressure = energy density = 0).

In the case of the Λ-term and pure empty space no non-spherical solutions were found, i.e. the spacetimes considered here are forced by the field equations to be spherically symmetric in the well known traditional sense. However, in the case of a nontrivial perfect fluid one solution is found which is not spherically symmetric as a spacetime. The reason of its nonsphericity is found to be the curvature of the lines onto which the 3-spaces are stringed (see section 5). It is the solution found in 1967 by Stephani [1].

This result can be explained intuitively in another way. A spacetime inside matter is described by a nonzero energy-momentum tensor which can be defined by its invariants (e.g. projections onto fixed directions). If the spacetime is not fully spherically symmetric (in the traditional sense), then at least one of the invariants should be not-spherically - symmetric in the ordinary sense. This invariant, then, clearly indicates the nonsphericity of the spacetime. The nontrivial fact shown in this paper is that such a solution exists. In the empty space, on the other hand, no material invariants are available, and so the nonsphericity is not easy to be revealed. In fact, as the paper shows, it cannot be revealed at all: the spacetime is then truly spherically symmetric.

The physical difference between the spacetime which is O(3)-symmetric as a whole and one that has only O(3)-symmetric hypersurfaces can be described in the following way.

If we are given the 3-spaces $t=$ const without any device to measure the time in different points, then we can reveal only the spherical symmetry of each 3-space by purely geometrical measurements. If, in addition, we attach a clock to each point of the space, then we can say: the spacetime considered here is spherically symmetric as a whole if its 3-spaces t = const are spherically symmetric and all the cloks placed on one sphere go at the same rate [2]. The solution from sec.5 fits this definition.

The plan of the paper is as follows. In sec.2 the problem is posed by writing a metric form concordant with all our assumptions. In sec. 3 we discuss the case which is analogous to the solution of Nariai [3] of ordinary spherical symmetry, and in fact we only recover the Nariai's solution. In sec.4 we discuss the case strictly analogous to the standard spherically symmetric spacetime and we find one non traditional solution: the one of Stephani [1]. The geometrical properties of the Stephani's solution are investigated in sec.5. In section 6 we discuss the most general spacetime obeying all the aforementioned assumptions and we show that sections 3 and 4 actually exhausted the problem.

The final sections of the paper are devoted to an analogous problem with homogeneity. Here it is assumed that the 3-spaces t = const are homogeneous with respect to a 3-parameter group acting transitively, but the whole spacetime is not necessarily invariant with respect to this group. The problem here is considerably more complicated, so it is assumed for simplicity that the 3-spaces are spherically symmetric in addition to being homogeneous, that they are stringed onto a line which is orthogonal to them all, and, as before, that matter displays no transversal motions to the distinguished observers.

One class of solutions, found previously by Stephani [1], was reobtained. Another class was also investigated, in which no new solutions were found explicitly. In both classes the geometry of each of the subspaces $t = \text{const}$ is the same as in the R-W metrics, but the curvature of the 3-spaces $t = \text{const}$ is varying in time in a different way, so that it can change its sign.

The present paper is one of the possible specifications of the idea of C.B.Collins [4] who proposed to investigate spacetimes having subspaces with definite symmetry groups. This line of research is meant to replace the nearly exploited line of investigating spacetimes which had definite symmetries in total. In fact, the present paper was motivated by the recent beautiful and enlightening criticism of standard cosmology by G.F.R.Ellis, expressed particularly in [5].

The calculations for this paper were carried out with use of the symbolic formula - manipulation computer system ORTOCARTAN [6,7]. With it, calculating the left-hand side of the Einstein field equations, a traditional nightmare of every relativist, has become a leisure time. Indeed, time has come to transfer this work totally to computers and save many people large parts of their lives.

2. Definition and statement of the purpose

As stated in the introduction we shall deal with such spacetimes, whose subspaces $t = \text{const}$ are all spherically symmetric in the normal sense. Thus, it should be possible to choose the coordinates in the spacetime so that in every space $t = \text{const}$ the metric form is:

$$ds_3^2 = \tilde{\gamma}(r)\, dr^2 + \tilde{\delta}(r)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \qquad (2.1)$$

where $\tilde{\gamma}(r)$ and $\tilde{\delta}(r)$ are arbitrary functions of the coordinate r [8], implicitly understood to be the values, at a fixed t, of some arbitrary functions of two variables, $\gamma^2(t,r)$ and $\delta^2(t,r)$ respectively.

The most general such spacetime has the metric form:

$$ds^2 = D^2(t,r,\vartheta,\varphi)dt^2 + 2\alpha_1(t,r,\vartheta,\varphi)dt\, dr + 2\alpha_2(t,r,\vartheta,\varphi)dt d\vartheta + 2\alpha_3(t,r,\vartheta,\varphi)dt\, d\varphi - \gamma^2(t,r)dr^2 - \delta^2(t,r)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \qquad (2.2)$$

where D, α_1, α_2 and α_3 are arbitrary functions of four variables. For simplicity, just to gain an insight into a new kind of geometry which seems not to have been considered before [9], we shall assume throughout the paper that the spaces (2.1) form the spacetime (2.2) by being stringed onto a timelike congruence which is orthogonal to them all, i.e. the t-lines are orthogonal to the spaces t = const. Consequently we assume that:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0. \qquad (2.3)$$

We shall also denote:

$$\gamma \stackrel{\text{def}}{=} e^{\mu(t,r)} \qquad (2.4)$$

The paper will be mostly devoted to the question: what kind of a source can generate the metric (2.2) - (2.3) through the Einstein field equations if the metric is not to be spherically symmetric in the traditional sense (i.e. the function D is to depend on at least one of the variables ϑ and φ).

The components of the Ricci and Einstein tensors, referred to in the paper, will be all with respect to the simplest orthonormal tetrad of differential forms connected with (2.2) - (2.3).

3. The Nariai-like case

In standard textbooks one usually chooses one of the coordinates (t,r) so that the function $\delta(t,r)$ in (2.2) assumes a pre-specified shape. Clearly, this is impossible when $\delta = \text{const}$, and we shall consider this case first. Thus we denote $\delta = L$ and consider the field equations for the metric:

$$ds^2 = (D\ dt)^2 - (e^{\mu}dr)^2 - (L d\vartheta)^2 - (L\ \sin\vartheta\ d\varphi)^2. \qquad (3.1)$$

If, as we assumed, the source in the field equations is no more general than a perfect fluid, then the energy-momentum tensor is of the form:

$$T^{ij} = (\epsilon + p)\, u^i u^j - p g^{ij}\ , \qquad (3.2)$$

where indices refer to the orthonormal tetrad. With our another assumption, that the velocity field may have only the u^t and u^r components, the Einstein field equations imply for the Ricci tensor:

$$R_{03} = R_{03} = R_{12} = R_{13} = R_{23} = R_{22} - R_{33} = 0. \qquad (3.3)$$

However, in the case of (3.1), $R_{01} = 0$ identically what means that the fluid must be moving along the t-lines, i.e. $u^r = 0$. The equations $R_{02} = R_{03} = 0$ say, respectively:

$$\mu_{,t}\ D_{,\vartheta}\ /LD^2 = 0, \qquad (3.4)$$

$$\mu_{,t}\ D_{,\varphi}\ /LD^2\ \sin\vartheta = 0.$$

Consequently, either $D_{,\vartheta} = D_{,\varphi} = 0$, or $\mu_{,t} = 0$. The first case leads to an ordinary spherically symmetric solution. Moreover, if $D_{,\vartheta} = D_{,\varphi} = 0$, then $R_{22} = R_{33} = L^{-2} = \text{const}$, $R_{00} + R_{11} = 0$, and so the source may only be the Λ - term, $\Lambda = L^{-2}$. This case was shown in [8] to lead to the solution of Nariai [3] , and so we can expect no new result along this line. Let us recall that the solution of Nariai has the structure of the cartesian product of a hyperboloid by a sphere, and is precisely the solution that is lost if one forgets to consider separately the case when the function δ in (2.2) is constant. Unfortumately, this is done in most textbooks. The metric of the Nariai's solution has the form:

$$ds^2 = \left[a(t)\cos(\ln\frac{r}{L}) + b(t)\sin(\ln\frac{r}{L})\right]^2 dt^2 + \\ - L^2(dr^2/r^2 + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \tag{3.5}$$

where a and b are arbitrary functions of time. Since $\Lambda = L^{-2}$, the solution has no proper limit $\Lambda \to 0$.

When we follow the other case in (3.4) we have from $R_{12} = R_{13} = 0$:

$$D_{,r\vartheta} = D_{,r\varphi} = 0. \tag{3.6}$$

Consequently:

$$D = F_1(t,\vartheta,\varphi) + F_2(t,r) \tag{3.7}$$

where F_1 and F_2 are new unknown functions. Now, from $R_{23} = 0$ we have:

$$- D_{,\vartheta\vartheta} + \cot\vartheta\, D_{,\varphi} = 0 \tag{3.8}$$

which means $F_1 = F_3(t,\varphi)\sin\vartheta + F_4(t,\vartheta)$. Next, from $R_{33} - R_{22} = 0$ we have:

$$D_{,\vartheta\vartheta} - D_{,\varphi\varphi}/\sin^2\vartheta - \cot\vartheta\, D_{,\vartheta} = 0 \tag{3.9}$$

which yields:

$$D = F_2(t,r) + A(t)\sin\vartheta\cos\varphi + B(t)\sin\vartheta\sin\varphi + \\ + C(t)\cos\vartheta. \tag{3.10}$$

The equation $R_{33} - R_{11} = 0$ now yields $e^{-2\mu}(D_{,rr} - \mu_{,r}D_{,r}) + \frac{1}{L^2}(D - D_{,\varphi\varphi}/\sin^2\vartheta - \cot\vartheta\, D_{,\vartheta}) = 0$, and this results in $A = B = C = 0$, $D = F_2(t,r)$. So finally we could not escape the full spherical symmetry, and we have reobtained the solution of Nariai [3] here.

4. The Schwarzschild-like case

From now on we shall consider only such metrics (2.2)-(2.3) in which δ is not constant. In this section we shall investigate the (apparently) special case when the function δ may be chosen to be the r-coordinate. In the standard spherically symmetric case the necessary and sufficient condition for this was the gradient of δ being a spacelike vector. There, however, we could employ transformations mixing t with r, i.e. $t = t(t', r')$, $r = r(t', r')$. Not so here: if D depends on ϑ or φ, then to preserve the form (2.2) with $\alpha_i = 0$ we may employ only the non-mixing transformations $t = t(t')$, $r = r(r')$. Consequently, δ may be chosen as the new coordinate r only when $\delta = \delta(r)$ (i.e. $\delta_{,t} = 0$) in the old coordinates.

The equations $R_{02} = R_{03} = 0$ are here nearly identical with (3.4) (only r replaces L) and so pose the same dilemma: either we return to the well-investigated case of ordinary spherical symmetry (see e.g. [10]), or $\mu_{,t} = 0$. We shall consider only the second case. Since however:

$$R_{01} = 2\mu_{,t}/ r D e^{\mu}, \tag{4.1}$$

we see that with $\mu_{,t} = 0$ we have $R_{01} = 0$, and so if the source is a perfect fluid, then it must move along the t-lines.

The equations $R_{12} = R_{13} = 0$ yield now:

$$- (rD_{,r\vartheta} - D_{,\vartheta})/r^2 D e^{\mu} = 0, \tag{4.2}$$

$$- (rD_{,r\varphi} - D_{,\varphi})/r^2 D e^{\mu} \sin\vartheta = 0,$$

and so:

$$D = F_1(t,\vartheta,\varphi)\, r + F_2(t,r). \tag{4.3}$$

The equations $R_{23} = R_{33} - R_{22} = 0$ yield, respectively, (3.8) and (3.9), and so:

$$D = r\left[A(t)\sin\vartheta\cos\varphi + B(t)\sin\vartheta\sin\varphi + C(t)\cos\vartheta\right] + F_2(t,r). \tag{4.4}$$

Now, $R_{11} - R_{22} = 0$ yields:

$$e^{-2\mu}\left[-D_{,rr} + (\mu_{,r} + \frac{1}{r})D_{,r} + \mu_{,r}D/r + D/r^2\right] + (D_{,\vartheta\vartheta} - D)/r^2 = 0. \tag{4.5}$$

Substituting (4.4) here we obtain the result that either $A = B = C = 0$, or:

$$e^{-2\mu}(\mu_{,r} + \frac{1}{r}) - \frac{1}{r} = 0. \tag{4.6}$$

Again, the first case is not interesting, being just spherically symmetric, so we shall consider only (4.6). Then, remembering that $\mu_{,t} = 0$, we obtain:

$$e^{-2\mu} = 1 + Kr^2 \tag{4.7}$$

where $K = \text{const.}$ Still, we have one more equation resulting from (4.5):

$$e^{-2\mu}\left[-F_{2,rr} + (\mu_{,r} + \frac{1}{r})F_{2,r}\right] = 0. \tag{4.8}$$

This has the solution:

$$F_2 = E(t)(1 + Kr^2)^{1/2} + s \tag{4.9}$$

where E is an arbitrary function of t. The quantity s, though in general being an arbitrary function of t, if it is not zero, may always be set equal to 1 by the coordinate transformation $\int s(t)dt = t'.$ Therefore we shall assume that $s = 1$ or $s = 0$.

Disregarding the equation of state, we have already fulfilled all the equations for a perfect fluid. The remaining ones just define the energy density and the pressure. When

(4.9) and (4.4) are substituted into the Ricci tensor, then the only nonzero tetrad components appear to be:

$$R_{00} = 3K\,(1 - s/D\,), \tag{4.10}$$

$$R_{11} = R_{22} = R_{33} = -\,3K(1 - s/3D).$$

From here we obtain for the energy density ϵ and the pressure p:

$$\varkappa\,\epsilon = -\,3K, \tag{4.11}$$

$$\varkappa\,p = 3K(1 - 2s/3D),$$

where $\varkappa = \frac{8\pi G}{c^4}$. Thus ϵ = const, and so the fluid is incompressible. Moreover, the metric corresponding to (4.10) is conformally flat which altogether strongly resembles the so - called interior Schwarzschild solution [11]. This solution was found by Stephani [1] as one of the metrics which can be imbedded in a flat 5-dimensional space.

5. Geometrical properties of the Stephani's solution

Let us display the solution explicitly:

$$ds^2 = D^2dt^2 - (1 + Kr^2)^{-1}\,dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2),$$

$$D = r\big[A(t)\sin\vartheta\,\cos\varphi + B(t)\sin\vartheta\sin\varphi + C(t)\cos\vartheta\big] + \tag{5.1}$$

$$+\,E(t)(1 + Kr^2)^{1/2} + s\,,$$

$s = 1$ or 0.

We see from (4.11) that the pressure differs from $(-\epsilon)$ (i.e. from the cosmological constant) by a term proportional to s, and so with $s = 0$ the source of the metric (5.1) reduces to the Λ - term, $\Lambda = 3K$. Then (5.1) seems at first sight to be a highly nonsymmetric generalization of the de Sitter solution [12] which results from (5.1) in its standard from when $A = B = C = s = 0$, $E = 1$. However, as stated above, the Weyl tensor of (5.1) is equal to zero.

Therefore, if in addition the Ricci tensor of (5.1) happens to be equal to the Λ - term, the spacetime becomes just a space of constant curvature with the Riemann, tensor given by:

$$R^{ij}{}_{kl} = K(\delta^i{}_k\, \delta^j{}_l - \delta^i{}_l\, \delta^j{}_k) \tag{5.2}$$

where the deltas are the Kronecker symbols. Thus in every invariant respect the metric (5.1) with $s = 0$ is equivalent to the de Sitter solution having the largest possible symmetry group. Moreover, with $K = 0$ the metric (5.1) is just flat. To demonstrate these facts by a direct coordinate transformation is a challenging exercise which the author did not undertake.

With $s \neq 0 \neq K$, and the functions A, B, C, E completely arbitrary the solution (5.1) has no symmetries.

The actual (when $s \neq 0 \neq K$) and spurious (when $s = 0$) lack of symmetry in (5.1) can be easily interpreted geometrically. Let us calculate the expression:

$$\frac{dt^\alpha}{d\lambda} + \left\{ {\alpha \atop \beta\gamma} \right\} t^\beta\, t^\gamma \overset{\text{def}}{=} a^\alpha \tag{5.3}$$

for the metric (5.1), where $\left\{ {\alpha \atop \beta\gamma} \right\}$ are Christoffel symbols and $t^\alpha = \delta^\alpha{}_0$, i.e. t^α is a vector field tangent to the t-lines, constant in the λ - parametrization and in the coordinates presently used. It appears that $a^\alpha \propto t^\alpha$ if and only if $A = B = C = 0$ (when $K = 0$) or $A = B = C = E = 0$ (when $K \neq 0$). But if $a^\alpha \propto t^\alpha$, then by changing to another parametrization we may achieve $a^\alpha = 0$, i.e. t^α is then a geodesic vector, i.e. the t-lines are then geodesics. Thus the functions $A(t)$, $B(t)$, $C(t)$ and $E(t)$ simply measure the geodesic curvature of the t-lines, and taking the limit $(A^2 + B^2 + C^2 + E^2) \longrightarrow 0$ corresponds to "staightening out" the t-lines onto which the spherical 3-spaces are stringed. In the case of the flat $(K = 0)$ or de Sitter $(K \neq 0 = s)$ spacetime, by straightening the t-lines we still remain within the same spacetime, while in the general case $(K \neq 0 \neq s)$ straightening the t-lines means changing the spacetime from the solution (5.1) to the interior Schwarzchild solution [11] which results from (5.1) when $A = B = C = E = 0$, $s \neq 0 \neq K$.

When the spacetime(5.1) is flat, $K = 0$, then the functions A, B and C have a very simple geometrical meaning. Namely, then we can change to the Cartesian-like coordinates $x = r \sin\vartheta \cos\varphi$, $y = r \sin\vartheta \sin\varphi$, $z = r \cos\vartheta$ to obtain:

$$ds^2 = D^2\, dt^2 - dx^2 - dy^2 - dz^2, \tag{5.4}$$

where:

$$D = A(t) \cdot x + B(t) \cdot y + C(t) \cdot z + E(t) + s \tag{5.5}$$

The first three terms is D remind of a scalar product of a 3-dimensional vector of components (A, B, C) by the "radius vector" (x, y, z). This "scalar product" is totally responsible for the spurious lack of symmetry. Indeed, A, B and C are then the spatial components of the vector $\frac{1}{D}\, a^{\alpha}$ where a^{α} is defined in (5.3), i.e. $a^x = AD$, $a^y = BD$, $a^z = CD$, and so they measure the components of the acceleration vector (5.3). Thus the 3-vector of components (A, B, C) points from a point P on a t-curve in the direction of the local center of curvature of the curve at P.

To summarize: we can string spherical 3-spaces onto orthogonal t-lines which are either: (I) "straight" (i.e. geodesic) or (II) "curved" (non-geodesic). If the spacetime so obtained appears empty (or endowed with the Λ-term only), then in both cases it is flat (or, respectively, of constant curvature). If there is any matter in the spacetime, then in case (I) the spacetime is fully spherically symmetric while in case (II) it is not. This altogether shows that the analogies with 2-dimensional surfaces from sec.1 were not quite superficial.

6. The general case with $\delta \neq$ const

Here the equations $R_{02}=R_{03}=0$ say, respectively:

$$D_{,\vartheta}(\delta_{,t}/\delta+\mu_{,t})/\delta D^2=0,$$
$$D_{,\varphi}(\delta_{,t}/\delta+\mu_{,t})/\delta D^2\sin\vartheta=0. \tag{6.1}$$

Again, since we consider the case $D_{,\vartheta}=D_{,\varphi}=0$ to be uninteresting, we have from here $\delta_{,t}/\delta+\mu_{,t}=0$, i.e.

$$\delta=f(r)\,e^{-\mu} \tag{6.2}$$

where $f(r)$ is an orbitrary function.

The equations $R_{12}=R_{13}=0$ then say:

$$(D_{,\vartheta}\,\delta_{,r}/\delta-D_{,r\vartheta})/e^{\mu}\delta D=0,$$
$$(D_{,\varphi}\,\delta_{,r}/\delta-D_{,r\varphi})/e^{\mu}\delta D\sin\vartheta=0. \tag{6.3}$$

Consequently:

$$D=F_1(t,\vartheta,\varphi)\,\delta+F_2(t,r) \tag{6.4}$$

(note the close analogy with (4.3)). The equations $R_{23}=R_{33}-R_{22}=0$ are again identical with (3.8) and (3.9) and so yield:

$$D=[A(t)\sin\vartheta\cos\varphi+B(t)\sin\vartheta\sin\varphi+C(t)\cos\vartheta]\delta+$$
$$+F_2(t,r). \tag{6.5}$$

In this point the line of reasoning must split to consider two cases: either the fluid in the spacetime, if any is present, moves along the t-lines, or it moves off the t-lines. In both cases no new solution emerges, so we defer the details to the appendices A and B. However, a few points in the second case are of some importance, so we shall deal with them here.

In agreement with our assumptions from sec. 1, if the fluid moves off the t-lines, then its velocity field u^α may have only the u^t and u^r components. Then the tetrad components of the velocity field will be denoted by U and V, i.e.:

$$u^i \overset{\text{def}}{=} U\,\delta^i_{\,0} + V\,\delta^i_{\,1}\,, \tag{6.6}$$

where

$$U^2 - V^2 = 1. \tag{6.7}$$

The tetrad components of the Einstein field equations are now:

$$\begin{aligned} G_{00} &= \frac{\varkappa}{c^2}\left[(\epsilon + p)\,U^2 - p\right], \\ G_{01} &= \frac{\varkappa}{c^2}(\epsilon + p)\,UV\,, \\ G_{11} &= \frac{\varkappa}{c^2}\left[(\epsilon + p)V^2 + p\right], \\ G_{22} &= G_{33} = \frac{\varkappa}{c^2}\,p\,, \end{aligned} \tag{6.8}$$

where $\varkappa = \dfrac{8\pi G}{c^2}$.

It is easy to see now the following. The components G_{00}, G_{01} and G_{11} define algebraically the functions p, ϵ and U (V being eliminated by virtue of (6.7)). Since $G_{22} = G_{33}$ is fulfilled identically by D given in (6.5), the component G_{22} alone defines the function p, too. In order that p defined by G_{22} be the same as p defined by G_{00}, G_{01} and G_{11}, the following equation must be fulfilled:

$$(G_{00} + G_{22})(G_{11} - G_{22}) - G_{01}^{\,2} = 0 \qquad (6.9)$$

This is the only differential equation that remains. Others are just algebraic definitions of ϵ, p, U and V.

Luckily, the equation (6.9) is solvable, but unfortunately no new solutions are contained in it: all of them are either spherically symmetric ($D,_{\vartheta} = D,_{\varphi} = 0$) or are contained in the class considered in sec.4. This is shown in the Appendix B.

In this way we conclude that all the solutions of our problem were exhausted in sections 3 and 4. Incidentally, this means that no generality was lost in sec.4 on assuming $\delta = r$. Here, however, this fact is a hard-calculation result of the field equations, as opposed to the ordinary spherically symmetric case, where $\delta = r$ (if δ not constant) was merely a choice of coordinates.

7. Spacetimes with homogeneous hypersurfaces

In analogy with the foregoing part of the paper we can consider spacetimes which are composed of homogeneous 3-spaces t = const orthogonally stacked, but which are not themselves invariant under the groups of symmetry of the 3-spaces. An ambitious project here would be to consider all the possible Bianchi types of transitive groups [13]. For simplicity we shall consider only the spacetimes which directly correspond to the standard Robertson-Walker Universes, and this special case will appear complicated enough for the beginning.

We shall thus deal with spacetimes in which the 3-spaces t = const are homogeneous and isotropic, i.e. are the same as in the R-W models.

Two inequivalent extensions of this kind were found, corresponding to two different representations of the R-W metrics. (The number of inequivalent extensions may be even larger corresponding to the different coordinates used to represent the R-W metrics. Because each 3-space $t=const$ is isotropic and homogeneous, there is no way to identify single points of it, and consequently there exists a multitude of correspondences between points of different 3-spaces, established by the family of the t-lines. Quite a different thing occurred in sections 1-6: a 3-space which is spherically symmetric but inhomogeneous has a well-defined center, and so it is most natural to assume, as we did, that one of the t-lines joins the centers of all the 3-spaces). We shall begin with the spacetime in which the 3-spaces $t=const$ have the metric [14] :

$$ds_3{}^2 = R^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right)\right] \qquad (7.1)$$

where R and k are constants, understood to be mementary values of certain functions $R(t)$ and $k(t)$ at a fixed t. For simplicity we shall again assume that the lines of the time coordinate are orthogonal to the spaces $t = const$ given by (7.1), and that the fluid source of the metric is moving, with respect to the t-lines, only radially if at all. Consequently, we assume the metric of the spacetime to be of the form:

$$ds^2 = D^2(t,r,\vartheta,\varphi)dt^2 - R^2(t)\left[\frac{dr^2}{1-k(t)r^2} + \right.$$

$$\left. + r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right)\right] \qquad (7.2)$$

where D is an arbitrary function of four variables while R and k are functions of t.

Since we have shown in sections 4 and 6 that δ (equal to rR here) may depend on t only when $D_{,\vartheta} = D_{,\varphi} = 0$, we will not be surprised to find in the next section that $D_{,\vartheta} = D_{,\varphi} = 0$ unless $R=$ const, $k=$const. However, one interesting new possibility arises here: if k indeed depends on t, and is positive for some t and negative for others, then the curvature of the 3-spaces $t =$ const, equal to k/R^2, may change its sign for a certain t. Thus, potentially, the closed model could evolve here into an open one or vice versa. In itself, the idea of making k a function of t is so simple that the question arises why nobody has thought of it before. The answer is: whenever spatially homogeneous cosmological models were considered, it was always assumed that the group acting transitively in the spaces $t =$ const is at the same time the symmetry group (or subgroup) of the whole spacetime. Under this assumption one concludes (see e.g. [15]) that the metric components may depend on t only through linear combinations of one-argument functions of t, and so if (7.2) is to be homogeneous in the traditional Bianchi sense, then only R may depend on time while k must be a constant.

8. General constraints from the field equations

Here we shall be interested only in the perfect fluid or dust as possible sources, and again the velocity field of the source, u^α, will be such that $u^\vartheta = u^\varphi = 0$. Consequently, the tetrad components of the Einstein tensor G_{ij}, in the orthonormal tetrad connected with (7.2), must fulfil the equations:

$$G_{02} = G_{03} = G_{12} = G_{13} = G_{23} = G_{22} - G_{33} = 0. \qquad (8.1)$$

The equations $G_{02} = G_{03} = 0$ yield, respectively:

$$\frac{1}{2}\alpha D_{,\vartheta}/rD^2R = 0,$$
$$\frac{1}{2}\alpha D_{,\varphi}/rD^2\sin\vartheta = 0, \tag{8.2}$$

where:

$$\alpha = r^2k_{,t}/(1-kr^2) + 4R_{,t}/R. \tag{8.3}$$

Thus either $D_{,\vartheta} = D_{,\varphi} = 0$ or $\alpha = 0$. We would tend to discard the first case as spherically symmetric and thus uninteresting. However, it is easy to see that $\alpha = 0$ implies $k_{,t} = R_{,t} = 0$, and so, on rescaling r by $r' = Rr$ and redefining k by $k' = -k/R^2$ we recover precisely the case considered in section 4. On the other hand, when $D_{,\vartheta} = D_{,\varphi} = 0$ we still retain the possibility of varying sign of curvature which seems interesting.

Therefore we consider the first case, thus returning to spherical symmetry rather soon. In this case the other equations of (8.1) are fulfilled identically.

9. The fluid moving along the t-lines

In this case the equation $G_{01} = 0$ must be fulfilled which says:

$$\left[rk_{,t}D/(1-kr^2) + 2D_{,r}R_{,t}/R\right](1-kr^2)^{1/2}/RD^2 = 0. \tag{9.1}$$

If $R_{,t} = 0$ then $k_{,t} = 0$, and so the case of sec. 4 is recovered. We shall then assume that $R_{,t} \neq 0$. Then, if $k_{,t} = 0$ we have from (9.1) $D_{,r} = 0$, i.e. $D = D(t)$. In this case D may by set equal to 1 by a coordinate transformation, and the standard Robertson-Walker metric is recovered. We shall be interested only in the new situation which results when $k_{,t} \neq 0 \neq R_{,t}\, D_{,r}$. Then, from (9.1):

$$D = \phi(t)\exp\left\{\left[R\, k_{,t} \ln(1-kr^2)\right] / 4\, kR_{,t}\right\}, \tag{9.2}$$

where $\phi(t)$ is an arbitrary function which may be set equal to 1 by a coordinate transformation. Then:

$$D = (1 - kr^2)^{\beta}, \text{ where:} \tag{9.3}$$

$$\beta = Rk_{,t}/4kR_{,t}. \tag{9.4}$$

Unfortunately, this is a blind alley. Further, rather involved equations which result on substituting (9.3) and (9.4) into the Einstein tensor, run into a contradiction unless $k_{,t} = 0$. Thus, only standard Robertson-Walker solutions are recovered here.

10. The general fluid

If we allow the fluid to move in any direction in the (t,r) space, then:

1. If it is a geniune perfect fluid, then the only equation to be obeyed by D is (6.9). After (6.9) is fulfilled, the pressure, energy density and velocity components are algebraically determined through G_{ij}.

2. If it is a dust, then the two equations:

$$G_{00}G_{11} - G_{01}^2 = 0, \tag{10.1}$$

$$G_{22} = 0 \tag{10.2}$$

must be fulfilled (note that, owing to spherical symmetry, $G_{22} = G_{33}$ is an identity).

No answer has been reached whether (10.1) - (10.2) can be fulfilled. The provisional answer should be "no" since then the function $D(t,r)$ must simultaneously obey two differential equations. The equation (6.9), on the other hand, can be fulfilled, being just one equation for one function

$D(t,r)$, with $k(t)$ and $R(t)$ being arbitrary parameters. The prospects of obtaining a general integral of that equation are, however, rather dim owing to its complexity.

The components of the Einstein tensor for the metric (7.2), with $D_{,\vartheta} = D_{,\varphi} = 0$, are written in the Appendix C.

11. A different extension of the R-W models

Let us now repeat the reasoning of sections 7-10 for a spacetime which is composed of these same 3-spaces in a different way. Let us take a 3-space t-const from a Robertson-Watker metric in a spherical conformally flat representation [16]:

$$ds_3^{\,2} = \frac{R^2}{(1+\frac{1}{4}kr^2)^2}\left[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right] \qquad (11.1)$$

where again R and k are understood to be momentary values of certain functions $R(t)$ and $k(t)$ at a certain value of t. If the 3-spaces (11.1) are stringed onto a congruence of t-lines which are orthogonal to them, then the specetime metric is:

$$ds^2 = D^2(t,r,\vartheta,\varphi)dt^2 - \left\{R^2(t)/\left[1 + \tfrac{1}{4}k(t)r^2\right]^2\right\}\left[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]. \qquad (11.2)$$

With k=const, (11.2) is just a different coordinate representation of the metric (7.2). With $k_{,t} \neq 0$, (11.2) in general <u>is not</u> a transform of (7.2) even if $D=D(t,r)$. This can be seen as follows. Let us denote R and k from (11.2) by $\mathfrak{R}$ and $\mathfrak{K}$ respectively and (t,r) by (t',r'). Then r' is related by r from (7.2) by:

$$r = \frac{\mathfrak{R}(t')}{R(t)} \cdot \frac{r'}{1 + \frac{1}{4}\mathfrak{K}(t')r'^2} \qquad (11.3)$$

Since $r = r(t', r')$, t from (7.2) must be transformed by $t = F(t', r')$, where F should be such that $g_{t'r'} = 0$, $g_{r'r'} = -\mathcal{R}^2(t')/[1 + \frac{1}{4}\mathcal{K}(t')r'^2]^2$.

This, however, means that F should simultaneously obey two differential equations which is possible only in special cases. Actually, the integrability condition $F_{,t'r'} = F_{,r't'}$ puts a first order equation on D, which will not be investigated here.

So in (11.2) we have a new class of spacetimes. We shall investigate them with the same assumptions concerning the sources as in previous cases.

The field equations $G_{02} = G_{03} = 0$ give the results:

$$\phi D_{,\vartheta} = \phi D_{,\varphi} = 0, \tag{11.4}$$

where

$$\phi := \left[-\frac{1}{2} r\mathring{k}R + 2\mathring{R}\,(1 + \frac{1}{4}kr^2)/r\right]/R^2D^2. \tag{11.5}$$

Thus either $D_{,\vartheta} = D_{,\varphi} = 0$ or $\phi = 0$. We see easily that $\phi = 0$ implies $\mathring{k} = \mathring{R} = 0$, and in this case the solution of section 4 will be reobtained. Consequently, only the case $D_{,\vartheta} = D_{,\varphi} = 0$ is worth investigating.

If $D_{,\vartheta} = D_{,\varphi} = 0$, then the equations $G_{12} = G_{13} = G_{23} = G_{22} - G_{33} = 0$ are all fulfilled identically. However, we have to consider separately the case of the fluid moving along the t-lines and the case of the fluid moving off the t-lines.

If the fluid moves along the t-lines, then the equation $G_{01} = 0$ must be fulfilled which reads:

$$(r - \frac{1}{4}kr^3/V)\mathring{k}/RD + (2V\mathring{R}/R^2 - \frac{1}{2}\mathring{k}r^2/R)D_{,r}/D^2 = 0 \tag{11.6}$$

where:

$$V := 1 + \frac{1}{4}k(t)r^2. \tag{11.7}$$

The equation (11.6) may be rewritten as:

$$\frac{2V}{R}\left(\frac{V_{,tr}}{VD} - \frac{V_{,t}V_{,r}}{V^2 D} - \frac{V_{,t}D_{,r}}{D^2} + \frac{\dot{R}}{R}\frac{D_{,r}}{D^2}\right) = 0, \qquad (11.8)$$

and this is easily integrated to yield $(V_{,t}/V - \dot{R}/R)/D =$

$= 1/F(t)$, i.e.:

$$D = F(t)\left(\frac{V_{,t}}{V} - \frac{\dot{R}}{R}\right) \qquad (11.9)$$

where F is an arbitrary function of t. With such D, the equation $G_{11} = G_{22}$ is obeyed automatically, and so the energy-momentum tensor defined by the field equations has the algebraic form necessary for a perfect fluid.

The density ϵ and the pressure p are given by:

$$\varkappa\epsilon = 3\left(\frac{k}{R^2} + \frac{1}{F^2}\right), \qquad (11.10)$$

$$\varkappa p = -\frac{k}{R^2} - \frac{3}{F^2} - 2\frac{\dot{F}}{F^2 D} + \frac{F\dot{k}}{R^2 D}\left(1 - \frac{1}{2}\frac{kr^2}{V}\right). \qquad (11.11)$$

More elegant formulae result if one parametrizes the functions after Stephani [1] as:

$$F = 1/\alpha, \quad V = \vartheta/a, \quad k = (C^2 - \alpha^2)/a^2, \quad R = 1/a \qquad (11.12)$$

where α, a and C are new functions of t. Then:

$$D = \dot{\vartheta}/\alpha\vartheta, \qquad (11.13)$$

$$\varkappa\epsilon = 3C^2 \qquad (11.14)$$

$$\varkappa p = -3C^2 + 2C\dot{C}\,\vartheta/\dot{\vartheta}. \qquad (11.15)$$

The metric in this case is:

$$ds^2 = (\dot{\vartheta}/\alpha\vartheta)^2\, dt^2 - \vartheta^{-2}\left[dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right], \qquad (11.16)$$

with:

$$\vartheta = a + \frac{1}{4}(C^2 - \alpha^2)r^2/a. \qquad (11.17)$$

The solution given by (11.14) - (11.17) is a special case of another solution found by Stephani [1] as embeddible in a 5-dimensional flat space. We shall deal with that solution more closely in the next section. Let us note here only that all the functions of t are completely arbitrary, and so k may change its sign as many times as one wishes. So the sign of spatial curvature is not a fixed property of the models of this class, just as we have argued in section 7. The solution reduces to ordinary Robertson-Walker metrics in the special cases when $C^2 - \alpha^2 = \epsilon a^2$ where $\epsilon = +1, 0$ or -1.

If the fluid is not moving along the t-lines, then, in addition to the equations $G_{02} = G_{03} = G_{12} = G_{13} = G_{22} - G_{33} = 0$, used up to (11.5), the single equation (6.9) has to be fulfilled. Its solution should be a generalization of the Stephani's model. We leave this for separate investigation in a future work.

12. Generalized Friedman models with wandering center of symmetry

We shall present here the farthest extension of the Friedman-Robertson - Walker models along the line of section 11. In the previous section some rudimentary regularity of the spacetime was preserved because the t-lines mapping each 3-space into other 3-spaces were assumed to join the points which were "corresponding" in the following sense. Each of the 3-spaces t=const was homogeneous and so had no center of symmetry defined geometrically. However, the coordinate systems used in the 3-spaces had all their centers (origins), and it was silently assumed that if a t-line passes through the origin of coordinates in one 3-space, then it contains all the other origins of the other 3-spaces. This can be seen from the form (11.2). Such an assumption limited the class of metrics considered, and as a result the field equations forced the whole spacetime to be spherically symmetric. This limitation can be, however, relaxed and we shall do it here. We will now assume that the (coordinate) centers of symmetry of different 3-spaces are arbitrarily shifted with respect to each of the t-lines.

It will be more convenient now to represent the metric form of the 3-space in such coordinates which explicitly exhibit the arbitrary position of the origin of coordinate system. One possible form is:

$$ds_3^{\,2} = (R/V)^2\,(dx^2 + dy^2 + dz^2) \tag{12.1}$$

where:

$$V = 1 + \frac{1}{4}\,k\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right] \tag{12.2}$$

Here R, k, x_0, y_0 and z_0 are arbitrary constants. In section 11 we have assumed that R and k were values of the functions $R(t)$ and $k(t)$ at a fixed moment of time, while x_0, y_0 and z_0 were genuine constants removable by a coordinate transformation describing a simultaneous shift of the origin in all the 3-spaces. This assumption is by no means necessary: x_0, y_0 and z_0 may be as well assumed to be functions of time. The spacetime metric is then:

$$ds^2 = D^2(t,x,y,z)dt^2 - \left[R^2(t)/V^2(t,y,x,z)\right](dx^2 + dy^2 + dz^2), \tag{12.3}$$

where:

$$V(t,x,y,z) = 1 + \frac{1}{4}\,k(t)\left\{\left[x - x_0(t)\right]^2 + \left[y - y_0(t)\right]^2 + \left[z - z_0(t)\right]^2\right\}. \tag{12.4}$$

This spacetime is composed of the same 3-spaces as the spacetime (11.2), but in general, with arbitrary $x_0(t)$, $y_0(t)$ and $z_0(t)$, the coordinate origins of two different 3-spaces do not belong to the same t-line. As one proceeds from one 3-space to another, the origin wanders with respect to the t-lines. This characterization seems coordinate-dependent, but the spacetime (12.3) is different from (11.2) because the metric (12.3)-(12.4) is not spherically symmetric (in fact, with the functions x_0, y_0, z_0 being completely arbitrary, (12.3) has no symmetries at all, as Killing equations show).

The field equations for the metric (12.3)-(12.4) were solved by Stephani [1], and we shall call the solution the Stephani Universe. It is obtained by assuming that the source in the Einstein's equations is a perfect fluid moving along the t-lines. The field equations $G_{01} = G_{02} = G_{03} = 0$ imply then again (11.9), and with (11.9) the equations $G_{12}=G_{13} = G_{23}=G_{11} - G_{22} = G_{22} - G_{33} = 0$ are fulfilled identically. With Stephani's parametrization (11.12) the same formulae, (11.14)-(11.15), for the density and pressure result.

13. Some properties of the Stephani Universe

From the formula (11.15) one can conclude that $p = 0$ implies that k, x_0, y_0 and z_0 are constants, and so then $D = D(t)$ (from (11.9) and (12.4)). In this case a coordinate transformation makes D equal to 1, and so the old Friedman models are reobtained. Note that this means that vanishing pressure forbids the sign of spatial curvature to vary with time. (The same conclusion was hinted to, by counting the number of equations and the number of unknown functions, in section 10). This statement can be supported by the following intuitive argument. In the normal, R-W cosmology, the behavior of the closed and the open Universe can be explained in purely Newtonian terms. The Universe is closed when its mass-density relative to the rate of expansion is large enough to halt the expansion by its gravitational field, and is open otherwise. Therefore, to have the closed Universe change into an open one a mechanism for the decay of mass would have to be invoked. In principle, one mechanism is conceivable. It is well-know that the pressure in a gas or fluid, which in Newtonian physics can only tend to expand the volume of the medium, gives, in the general-relativity theory, a positive contribution to the energy-density [17], and so, at large values, exhibits the opposite tendency to enhance the self-gravitation.(1) In an expanding Universe the pressure would gradually diminish. It is therefore possible, in principle, that initially the contribution of pressure to self-gravitation would be large enough to close the Universe, while in the later stages of expansion this effect of pressure would become negligible and thus, with suitable amount of rest-mass, the Universe would open up. The opposite change would occur in a contracting Universe.

This intuitive argument does not explain how several changes, from positive curvature to negative and back to positive, and so forth, could occur during the evolution of the model. Such changes are mathematically evidently possible,

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(1) For the same amount of matter, larger pressure gradient is required to support a relativistic star in equilibrium than to support a newtonian star, and the increment in gradient is proportional to pressure - just as if pressure had its weight.

as is seen from (11.2) and (12.4): the Stephani Universes exists with an arbitrary function $k(t)$. The physics and astronomy in such a Universe will be investigated in separate papers. Let us note here only the obvious conclusion that, since the energy density is a function of time only, and the pressure depends also on the other coordinates, no equation of state of the simple form $\epsilon = \epsilon(p)$ is admissible.

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Appendix A

We shall consider here the first branch of the line of reasoning that splitted after (6.5): the one in which the fluid moves along the t-lines. Then the velocity field u^α has only the t-component, and so $R_{01} = 0$, which means:

$$2(\delta,_t D,_r - \delta,_{tr} D + \mu,_t \delta,_r D)/\delta D^2 e^\mu = 0. \qquad (A.1)$$

Substituting (6.5)here and equating to zero the coefficients of independent functions of ϑ and φ we obtain the result that either $A = B = C = 0$ or:

$$\mu,_{rt} - \mu,_r \mu,_t + f,_r \mu,_t/f = 0 \qquad (A.2)$$

where we have used the fact that $fe^{-\mu} = \delta$. The first case is again spherically symmetric, and so not interesting, while the second one yields:

$$\delta \equiv fe^{-\mu} = g(t) + h(r) \qquad (A.3)$$

where g and h are arbitrary functions. The equation (A.1) yields one more result:

$$F_2,_r \delta,_t - F_2 \delta,_{tr} + F_2 \mu,_t \delta,_r = 0, \qquad (A.4)$$

which, with use of (A.3), leads to:

$$[(g + h)F_{2,r} - F_2 h_{,r}]g_{,t}/f = 0. \tag{A.5}$$

Thus either $g_{,t} = 0$, or:

$$F_2 = E(t)(g + h). \tag{A.6}$$

When $g_{,t} = 0$ we have $\delta = \delta(r)$, and this case was considered in section 4. Consequently, we shall deal only with the new case, (A.6) with $g_{,t} \neq 0$. Then g may be chosen as the new t coordinate, so:

$$g = t. \tag{A.7}$$

The equation which was still not considered is $R_{11} - R_{22} = 0$. It says:

$$\phi D^3 + (E/\delta)D^2 + (2/\delta^2)D + 2A_{,t} \sin\vartheta \cos\varphi + \\ + 2B_{,t} \sin\vartheta \sin\varphi + 2C_{,t} \cos\vartheta = 0, \tag{A.8}$$

where:

$$\phi \overset{\text{def}}{=} -2/\delta^2 + 2\delta G_{,r}H_{,r}/H^3 - 2\delta G_{,rr}/H^2 = \phi(t,r). \tag{A.9}$$

With respect to the functions of ϑ and φ (A.8) is an algebraic equation (actually it is a polynomial with respect to $\sin\vartheta$, $\cos\vartheta$, $\sin\varphi$ and $\cos\varphi$, with coefficients depending on t and r). Consequently, the coefficients of independent functions of ϑ and φ should vanish separately. Substituting (6.5), (A.6) and (A.7) in (A.8), replacing every power of $\cos\vartheta$ higher than the first by use of $\cos^2\vartheta = 1 - \sin^2\vartheta$ (so that only $\cos\vartheta$ may remain), and similarly for $\cos\varphi$, and finally taking the coefficients of $\sin^3\vartheta \sin^3\varphi$ and $\sin^3\vartheta \sin^2\varphi \cos\varphi$, respectively, we obtain:

$$\delta^3 \phi (-3A^2B + B^3) = 0, \tag{A.10}$$

$$\delta^3 \phi (3AB^2 - A^3) = 0.$$

Let us suppose that $\phi \neq 0$. Then (A.10) imply at once that $A = B = 0$. Substituting this back into (6.5) and (A.8), and equating to zero the coefficient of $\sin^2\vartheta \cos\vartheta$ we obtain $\delta^3 \phi C^3 = 0$, i.e. $C = 0$. Thus when $\phi \neq 0$ we return to the case of ordinary spherical symmetry which we disregard here. The only interesting solution of (A.10) might thus be $\phi = 0$. This, however, implies $(\phi/\delta)_{,t} = 0$, and from (A.9) we see that then we would have $\delta_{,t} = 0$ what is impossible since $\delta = t + h(r)$.

In this way we have obtained a contradiction. Consequently, when δ is not constant, and the source in the field equations is no more general than a perfect fluid moving along the t-lines, then either δ is a function only of r (the case considered in section 4) or the metric is spherically symmetric in the ordinary sense.

Appendix B

Here we shall follow till the end the other line of reasoning that branched off after (6.5) which we followed in sec.6 till (6.9).

With D given by (6.5) and δ by (6.2) our metric may be written as:

$$ds^2 = (D dt)^2 - \left(\frac{f(r)dr}{\delta}\right)^2 - (\delta\, d\vartheta)^2 - (\delta \sin\vartheta\, d\varphi)^2 \qquad (B.1)$$

where:

$$D = \left[A(t)\sin\vartheta\cos\varphi + B(t)\sin\vartheta\sin\varphi + C(t)\cos\vartheta + F(t,r)\right]\delta(t,r) \qquad (B.2)$$

(we have denoted $F_2 = F\delta$). We see that by transforming the coordinate r to $r' = \int f(r)dr$ we can obtain $f = 1$ in the new coordinates.

The tetrad components of the Einstein tensor for the metric (B.1) (with f = 1) are:

$$G_{00} = \delta^{-2} - 3\,\delta_{,r}^{\;2} - 2\delta\delta_{,rr} - \delta_{,t}^{\;2}/\delta^2 D^2\,,$$

$$G_{01} = 2\delta\delta_{,t}F_{,r}/D^2 - 2\,\delta_{,tr}/D\,,$$

$$G_{11} = -3/\delta^2 + 3\,\delta_{,r}^{\;2} + 2F/\delta D + \delta_{,t}^{\;2}/\delta^2 D^2 - 2\,\delta_{,tt}/\delta D^2 +$$
$$+ 2\,\delta^2\delta_{,r}F_{,r}/D + 2C_{,t}\delta_{,t}\cos\vartheta/D^3 + 2A_{,t}\delta_{,t}\sin\vartheta\cos\varphi/D^3 +$$
$$+ 2B_{,t}\delta_{,t}\sin\vartheta\sin\varphi/D^3 + 2F_{,t}\delta_{,t}/D^3, \qquad (B.3)$$

$$G_{22} = G_{33} = -\delta^{-2} + 3\delta_{,r}^{\;2} + F/\delta D + 2\delta\delta_{,rr} - \delta_{,t}^{\;2}/\delta^2 D^2 +$$
$$+ 4\,\delta^2\delta_{,r}F_{,r}/D + \delta^3 F_{,r}/D.$$

When (B.3) are substituted into (6.9), and the result is multiplied by D^5, we obtain an equation of the form [polynomial in $\sin\vartheta$, $\cos\vartheta$, $\sin\varphi$, $\cos\varphi$] = 0, the coeficients of the polynomial being functions of t and r. When we replace every power of $\cos\vartheta$ and $\cos\varphi$ higher than the first with use of $\cos^2\vartheta = 1 - \sin^2\vartheta$, and the same for φ, so that $\cos\vartheta$ and $\cos\varphi$ appear only linearly if at all, then we may equate to zero each coefficient of the polynomial. The coefficients of $\sin^5\vartheta\,\sin^5\varphi$ and $\sin^5\vartheta\,\sin^4\varphi\,\cos\varphi$ yield respectively:

$$\psi\delta^5(-10A^2B^3 + 5A^4B + B^5) = 0, \qquad (B.4)$$
$$\psi\delta^5(5AB^4 - 10A^3B^2 + A^5) = 0,$$

where:

$$\psi = -3/\delta^4 - 9\delta_{,r}^{\;4} - 6\,\delta\delta_{,r}^{\;2}\delta_{,rr} + 12\,\delta_{,r}^{\;2}/\delta^2 + 6\,\delta_{,rr}/\delta. \qquad (B.5)$$

If $\psi \neq 0$, then $A = B = 0$. In this case, substituting $A=B=0$ back into (6.9) and taking the coefficient of $\sin^4\vartheta\,\cos\vartheta$ we obtain $\psi\,\delta^5C^5 = 0$, i.e. $C = 0$, so we return to ordinary spherical symmetry. Consequently, we shall consider only the case $\psi = 0$, i.e.:

$$3(1 - \delta^2 \delta_{,r}{}^2)(2\delta_{,rr}/\delta + 3\delta_{,r}{}^2/\delta^2 - 1/\delta^4) = 0. \qquad (B.6)$$

In this way the terms proportional to D^5 in (6.9) vanish. Taking next the terms proportional to D^4, and equating to zero the coefficients of $\sin^4\vartheta \sin^4\varphi$ and $\sin^4\vartheta \sin^3\varphi \cos\varphi$ we obtain:

$$\begin{aligned} &2\chi\delta^4(-6A^2B^2 + A^4 + B^4) = 0, \\ &2\chi\delta^4(4AB^3 - 4A^3B) = 0 \end{aligned} \qquad (B.7)$$

where:

$$\chi = (1/\delta^3 - 3\delta_{,r}{}^2/\delta)F - (2\delta_{,rr}/\delta + 3\delta_{,r}{}^2/\delta^2 + \\ - 1/\delta^4)\delta^4\delta_{,r}F_{,r} . \qquad (B.8)$$

Again, with $\chi = 0$ we obtain $A = B = 0$, and then soon $C = 0$, similarly as before. Consequently, the meaningful solution of (B.7) is for us $\chi = 0$. Eliminating now the second term in parentheses in (B.8) by multiplying it by $(1 - \delta^2 \delta_{,r}{}^2)$ and using (B.6) we obtain:

$$(1 - \delta^2 \delta_{,r}{}^2)(1 - 3\delta^2 \delta_{,r}{}^2)F/\delta^2 = 0. \qquad (B.9)$$

There are 3 cases to be considered in (B.9). When $(1 - \delta^2 \delta_{,r}{}^2) = 0$, then $\chi = 0$ implies $F = 0$. When $(1 - 3\delta^2\delta_{,r}{}^2) = 0$, then (B.6) is contradicted. So anyway we cannot escape the third case:

$$F = 0. \qquad (B.10)$$

From (B.6) we have now either $(1 - \delta^2 \delta_{,r}{}^2)$, or:

$$2\delta_{,rr}/\delta + 3\delta_{,r}{}^2/\delta^2 - 1/\delta^4 = 0. \qquad (B.11)$$

It is seen that (B.11) contains the former equation as a special case, so it is enough to consider (B.11) as the result of (B.6). It is easily integrated to give:

$$\delta^3 \delta_{,r}{}^2 - \delta = H(t), \tag{B.12}$$

where H is an arbitrary function.

With $F = 0$ and (B.12) fulfilled, the equation (6.9) simplifies to:

$$D^{-5}(D^3 \mathcal{D}_1 + \delta_{,t}{}^2 \mathcal{D}_2) = 0 \tag{B.13}$$

where:

$$\mathcal{D}_1 = 3\delta_{,t}{}^2/\delta^4 - 3\delta_{,t}{}^2\delta_{,r}{}^2/\delta^2 - 4\delta_{,tr}{}^2 , \tag{B.14}$$

$$\mathcal{D}_2 = D(-\delta_{,t}{}^2/\delta^4 + 2\delta_{,tt}/\delta^3) - 2(\delta_{,t}/\delta^2)(A_{,t}\sin\vartheta\cos\varphi + B_{,t}\sin\vartheta\sin\varphi + C_{,t}\cos\vartheta). \tag{B.15}$$

Analogously as before we conclude now from (B.13) that either $A = B = C = 0$ which case we disregard, or $\mathcal{D}_1 = 0$. Then, however, either $\delta_{,t} = 0$, and this case was considered in sec.4, or $\mathcal{D}_2 = 0$ what implies:

$$-A\delta_{,t}{}^2/\delta^3 + 2A\delta_{,tt}/\delta^2 - 2A_{,t}\,\delta_{,t}/\delta^2 = 0, \tag{B.16}$$

and two more equations resulting from (B.16) by substituting B and C, respectively, for A. They are easily integrated, to yield:

$$\delta = \left[K(r)\int A(t)dt + L(r)\right]^2 \tag{B.17}$$

where K and L are arbitrary functions. The other two equations yield $B = C_1 A$, $C = C_2 A$ where C_1 and C_2 are arbitrary constants.

When (B.17) is substituted in (B.11) we obtain that either $A = 0$, what we disregard because then also $B = C = 0 = D_{,\vartheta} = D_{,\varphi}$ or $K = 0$, what we disregard, too, as then $\delta_{,t} = 0$, or:

$$4K_{,r}{}^2 + K K_{,rr} = 0, \tag{B.18}$$

$$K^2L_{,rr} + 8KK_{,r}L_{,r} + KK_{,rr}L = 0.$$

The equations (B.18) are easily integrated, but then we remain with:

$$\left(K\int A\,dt + L\right)^6 \left(4L_{,r}^{\;2} + LL_{,rr}\right) = \frac{1}{4} \qquad \text{(B.19)}$$

which is the remnant of (B.11) after (B.18) are fulfilled. Now, if $A \neq 0 \neq K$, then (B.19) yields on differentiating with respect to t:

$$4L_{,r}^{\;2} + LL_{,rr} = 0 \qquad \text{(8.20)}$$

what evidently contradicts (B.19).

Thus in this branch we are also only able to recover the case of section 4 or to return to spherical symmetry.

Appendix C

We give here the tetrad components of the Einstein tensor for the metric (7.2). A dot denotes the t-derivative, a prime - the r-derivative.

$$G_{00} = 3k/R^2 + r^2\dot{k}\,\dot{R}/R(1 - kr^2)D^2 + 3\dot{R}^2/R^2D^2,$$

$$G_{01} = r\dot{k}/R(1 - kr^2)^{1/2}D + 2(1 - kr^2)^{1/2}\dot{R}D'/R^2D^2,$$

$$G_{11} = -2rkD'/R^2D - k/R^2 + 2D'/rR^2D - \dot{R}^2/R^2D^2 + 2\dot{R}\dot{D}/RD^3 - 2\ddot{R}/RD^2,$$

$$\begin{aligned} G_{22} = G_{33} = {} & -rkD'/R^2D - k/R^2 + (1 - kr^2)D'/rR^2D + \\ & - \tfrac{3}{2}r^2\dot{k}\dot{R}/R(1 - kr^2)D^2 + \tfrac{1}{2}r^2\dot{k}\dot{D}/(1 - kr^2)D^3 + \\ & - \tfrac{1}{2}r^2\ddot{k}/(1 - kr^2)D^2 - \tfrac{3}{2}r^4\dot{k}^2/(1 - kr^2)^2D^2 + \\ & - \dot{R}^2/R^2D^2 + (1 - kr^2)D''/R^2D + 2\dot{R}\dot{D}/RD^3 - 2\ddot{R}/RD^2. \end{aligned}$$

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