A NEWTONIAN MODEL OF THE SOURCE OF THE KERR METRIC

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A complete newtonian model of the source of the gravitational field whose equipotential surfaces are confocal revolution ellipsoids is presented. The solution exhibits a singularity familiar from the Kerr metric.

1. Introduction. In one of my earlier works [1] I argued that the gravitational field described by the Kerr metric should correspond to a newtonian potential whose level surfaces are oblate confocal revolution ellipsoids. In short, the argument was as follows: in the Kerr solution there exists a special timelike congruence of curves such that each 3-space locally orthogonal to a curve is composed of confocal oblate ellipsoids strung onto curves orthogonal to them all. The 3-space thus composed is itself curved, but the curvature is determined only by the mass of the source, and for zero mass the 3-space, along with the whole spacetime, becomes flat. It is then the euclidean space represented in spheroidal coordinates. (Ref. [1] contains also a review of other works where the connection of the Kerr metric with such a newtonian field was revealed by different methods.)

While only descriptive, this argument shows that the Kerr spacetime is composed of confocal oblate revolution ellipsoids in a way very similar to the way in which the Schwarzschild spacetime is composed of spheres. It is not clear what the geometrical and physical meaning is of these ellipsoids in the spacetime, but they are suggestive of equipotential surfaces of the gravitational field; all the more so because in the limit $a \to 0$ the ellipsoids degenerate into spheres of the Schwarzschild geometry. If such a connection really exists, then it would be useful to investigate the source of the newtonian gravitational field whose equipotential surfaces are confocal oblate revolution ellipsoids. Such a source, if obtainable from a solution of Einstein's equations by some kind of limiting procedure, would be a direct demonstration of the fact that a material source of the Kerr field exists.

The present paper shows how the desired newtonian solution can be found. In fact, one source of the field in question has been known for a long time. It is a shell of zero thickness and finite surface density of matter, the so called thin homoeoid, coinciding with one of the equipotential ellipsoids (see e.g. refs. [2,3], the solution was first found by Chasles in 1840). The new solution found here describes an extended perfect fluid body of ellipsoidal shape supported, through the Euler equations of motion, by spatial distributions of mass-density, pressure and local angular velocity of rotation. It is one of the few solutions of newtonian hydrodynamics which describe rotating fluid bodies of nonuniform density, and so might be valuable irrespective of its connection with the general theory of relativity.

2. The gravitational field outside the source. The exterior potential of the thin homoeoid of mass M, at the point with cartesian coordinates (X, Y, Z), is:

$$V_{\rm H}(X, Y, Z) = - \int_{\mathcal{V}} [(X - x)^2 + (Y - y)^2 + (Z - z)^2]^{-1/2} \frac{GM}{2\pi r_0 (r_0^2 + a^2)} \delta\left(\frac{x^2 + y^2}{r_0^2 + a^2} + \frac{z^2}{r_0^2} - 1\right) d_3 x , \qquad (2.1)$$

where (x, y, z) are cartesian coordinates of the current point of integration, r_0 is the minor semiaxis of the homoeoid,

a is its focal radius, and $\delta(\cdot)$ is the Dirac delta function centered on the homoeoid. The region of integration, $\mathcal{V}$ is any volume totally containing the homoeoid, and the delta function is multiplied by constant factors so that the integral yields the correct value of the mass parameter in the potential.

Let us transform eq. (2.1) to the spheroidal coordinates (r, ϑ, φ):

$$x = (r^2 + a^2)^{1/2} \sin \vartheta \cos \varphi, \quad y = (r^2 + a^2)^{1/2} \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta, \tag{2.2}$$

and similarly for $(X, Y, Z) \to (R, \Theta, \Phi)$. The equation $r = \text{const.}$ describes an oblate revolution ellipsoid, different ellipsoids being confocal. Then eq. (2.1) changes to

$$V_H(R, \Theta, \Phi) = -\int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \int_{f_1(\vartheta,\varphi)}^{f_2(\vartheta,\varphi)} dr (r^2 + a^2 \cos^2\vartheta) \sin \vartheta\, d^{-1} \frac{GM}{2\pi r_0 (r_0^2 + a^2)} \delta\left(\frac{(r_0^2 + a^2 \cos^2\vartheta)(r^2 - r_0^2)}{r_0^2(r_0^2 + a^2)}\right), \tag{2.3}$$

where $f_1(\vartheta, \varphi) < r_0 < f_2(\vartheta, \varphi)$ for all ϑ, φ; f_1 and f_2 being otherwise arbitrary, and:

$$d := [R^2 + r^2 - 2(R^2 + a^2)^{1/2}(r^2 + a^2)^{1/2} \sin \Theta \sin \vartheta \cos(\varphi - \Phi)$$

$$- 2Rr \cos \Theta \cos \vartheta + a^2 \sin^2\Theta + a^2 \sin^2\vartheta]^{1/2}. \tag{2.4}$$

Performing the integration with respect to r in eq. (2.3) we obtain

$$V_H(R, \Theta, \Phi) = -\frac{GM}{4\pi} \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \sin \vartheta / d \Big|_{r=r_0}. \tag{2.5}$$

The analytic proof that the field given by eq. (2.5) is independent of Θ follows from ref. [4] (it is performed there in a different coordinate system). That V_H is independent of Φ is easily seen. Consequently, V_H is a function of R only, its dependence on Θ and Φ being spurious.

Knowing this, it is easy to guess a continuous density distribution $\rho(r, \vartheta)$ to replace the δ-like distribution in eq. (2.3) in such a way that the resulting field depends still only on R. It is

$$\rho(r, \vartheta) = f(r)/(r^2 + a^2 \cos^2\vartheta), \tag{2.6}$$

where f is an arbitrary function of r. The exterior field of a body of this density distribution, bounded by the ellipsoid $r = r_0$, is

$$V_{\text{ext}}(R) = -G \int_0^{r_0} f(r)\, dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \sin \vartheta / d. \tag{2.7}$$

The integrand to be integrated with respect to ϑ and φ in eq. (2.7) is the same as in eq. (2.5), except that r_0 is replaced by r, and so the result of integration is, as before, independent of Θ and Φ. If so, we then substitute $\Theta = 0$ in eq. (2.7) to calculate

$$V_{\text{ext}} = -(GM/a) \arctan(a/R), \tag{2.8}$$

where

$$M = 4\pi \int_0^{r_0} f(r')\, dr'. \tag{2.9}$$

The field (2.8) can be verified to be the unique solution of the Laplace equation for a potential depending solely on R and obeying the boundary conditions $\lim_{R \to \infty} V = 0$, $\lim_{R \to \infty} (R^2\, dV/dR) = GM$.

3. The field inside the source. Let us now solve the Poisson equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\, V = 4\pi G\rho(x,y,z)\,, \tag{3.1}$$

assuming that the exterior surface of the body generating the potential V is an ellipsoid of revolution, and that V is constant on the ellipsoids confocal to the outer surface. If $V = V(r)$, then eq. (3.1) assumes the form

$$[(r^2 + a^2)\,\mathrm{d}^2 V/\mathrm{d}r^2 + 2r\,\mathrm{d}V/\mathrm{d}r]/(r^2 + a^2 \cos^2\vartheta) = 4\pi G\rho(r,\vartheta,\varphi)\,. \tag{3.2}$$

It is seen that eq. (3.2) may have a solution only when ρ is of the form (2.6). Solving eq. (3.2) with ρ given by eq. (2.6) we obtain

$$V_{\text{int}}(r) = \int_0^r \frac{4\pi G}{r'^2 + a^2}\, \mathrm{d}r' \int_0^{r'} f(r'')\,\mathrm{d}r'' + V(0) - aV'(0)\arctan(r/a)\,, \tag{3.3}$$

where $V(0)$ and $V'(0)$ are the values of V and $\mathrm{d}V/\mathrm{d}r$, respectively, at $r = 0$. In what follows we shall assume $V'(0) = 0$ while the value of $V(0)$ should be chosen so that $V_{\text{ext}}(r_0) = V_{\text{int}}(r_0)$.

4. Motions and stresses supporting the body. We shall now show that the source in question may be described as a portion of a rotating perfect fluid obeying the Euler equations of motion and endowed with a certain distribution of angular velocity $\omega(r,\vartheta)$. We assume that the oblateness of the body is caused by rotation around the body's minor diameter. Thus, in the spheroidal coordinates, the only nonzero component of the velocity field of matter is

$$\mathrm{d}\varphi/\mathrm{d}t = \omega(r,\vartheta)\,, \tag{4.1}$$

or, in cartesian coordinates, $\boldsymbol{\omega} = (0,0,\omega)$ is the vector of the angular velocity. Then the cartesian components of the velocity field are

$$v^x = \omega\,\partial x/\partial\varphi\,, \quad v^y = \omega\,\partial y/\partial\varphi\,, \quad v^z = 0\,. \tag{4.2}$$

We find now that the equation of continuity is fulfilled identically, while from the Euler equations of motion:

$$\frac{\partial v^i}{\partial t} + \sum_{j=1}^{3} v^j v^i{}_{,j} = -\rho^{-1} p_{,i} - V_{,i}\,, \tag{4.3}$$

we obtain for the x and y components:

$$\omega^2 = [r/f(r)]p_{,r} + [\cos\vartheta/f(r)\sin\vartheta]p_{,\vartheta} + GrM(r)/(r^2 + a^2)(r^2 + a^2\cos^2\vartheta) \tag{4.4}$$

and for the z component:

$$[(r^2 + a^2)/f(r)]\cos\vartheta\, p_{,r} - [r\sin\vartheta/f(r)]p_{,\vartheta} + G\cos\vartheta\, M(r)/(r^2 + a^2\cos^2\vartheta) = 0\,, \tag{4.5}$$

where $M(r)$ is given by eq. (2.9) with r_0 changed to r.

We see now that eq. (4.5) is a linear partial differential equation which determines p, and is solvable with any $f(r)$ and any boundary condition at $r = r_0$. After p is found from eq. (4.5), ω is simply calculated from eq. (4.4). The fact that $f(r)$ is quite arbitrary gives some place for a class of equations of state — a very desirable feature which will not be exploited here.

Eq. (4.5) may be solved by formal integration, but the solution gives little insight into the problem. It is given in the Appendix.

Let us note finally that $p = p(r)$ is not a solution of eq. (4.5), unless $a = 0$. This, together with eq. (2.6), shows that neither ρ nor p can be constant on all the surfaces $r = \text{const}$.

5. General properties of the configuration. Note from eq. (2.6) that $\rho_{,\vartheta} > 0$ for $0 < \vartheta < \pi/2$ and $\rho_{,\vartheta} < 0$ for $\pi/2 < \vartheta < \pi$, while, at a fixed r, for $\vartheta = 0$ and $\vartheta = \pi$ the density ρ attains its minima and for $\vartheta = \pi/2$ it attains a maximum. The variability of ρ with respect to r is, on the other hand, to a large extent arbitrary, owing to the arbitrariness of $f(r)$. Thus, for instance, one may assume an arbitrary distribution of $\rho(r)$ along the half-line $\vartheta = 0$, and every such configuration will obey the equations of continuity and of motion, though not necessarily the equation of state.

If we assume that $\rho_{,r} < 0$, then from the foregoing remarks it follows that the surfaces of constant ρ are more oblate than the surfaces of constant $V(r)$. Consequently, the equidensity surfaces cross the outer surface of the body along circles $\{r = r_0, \vartheta = \text{const.}\}$. Note however that if $\rho_{,r} < 0$ all the way from the surface to $r = 0$, then $\rho|_{r=0} > 0$. This means that $f(0) > 0$. But then at $\{r = 0, \vartheta = \pi/2\}$ ρ grows to infinity, i.e. a ring-like singularity on the focal ring of the ellipsoids appears. To prevent the singularity $f(r)$ would have to tend to zero for $r \to 0$ at least as rapidly as r^2. In this case, however, another singular behavior results: ρ is regular on the focal ring, but vanishes on the disk spanned by the ring, $\{r = 0, \vartheta < \pi/2\}$. This kind of singularity is perhaps less ridiculous, but then ρ cannot be monotonic with respect to r, unless it grows all the way from $r = 0$ to the surface. Consequently, either there is a singularity on the focal ring or ρ attains its maximum somewhere between the center and the surface, or right at the surface.

The ring-like singularity is familiar from the investigations of the Kerr metric source [5–7]: it appeared inescapably whenever one tried to assume that the source is the disk to which the ellipsoids shrink in the limit $r \to 0$. We see that this wild behavior is not specific to general relativity. Note, however, that the singularity appears only in the interior solution; the exterior potential (2.8) is nonsingular for all R, including $R = 0$, i.e. it is finite even if the source is squeezed to the ring.

Our model duplicates still one more property observed for the Kerr metric (see e.g. ref. [8]); that all the multipole moments of the gravitational field are determined by the same parameter a, and so are rigidly interlocked. We have seen that in spite of this the source is to a large extent flexible (arbitrary $f(r)$!).

Another curious property of our model is the fact that as we proceed to the interior of the body, the ellipsoids of constant potential become more and more oblate, and finally at $r = 0$ they degenerate into a disk of radius a. The outer surface of the body is one of the equipotential ellipsoids. Consequently, the smaller the body is (with fixed a), the more it is oblate.

The fluid in the body moves necessarily with shear (shear vanishes only when $\omega = \text{const.}$, but $\omega = \text{const.}$ is not a solution of eqs. (4.4), (4.5) unless $\omega = a = 0$). Thus our model can describe only nonviscous fluid bodies because the viscosity would damp the shear.

It is interesting to note that, in spite of all the aforementioned curiosities, in the limit $a \to 0$ our solution reproduces *all* the possible spherically symmetric configurations, and not just some of them. Moreover, it shares some interesting properties with spherically symmetric configurations, like e.g. the exterior field in a given point of space, with a fixed, being determined solely by the mass parameter M, and being independent of the size of the source. Also, in such a configuration, the gravitational force at an interior point P, exerted by the shell lying outside the ellipsoid E which passes through P and is confocal to the outer surface, is zero. The gravitational force at P is determined solely by the mass inside E. This can be seen from eqs. (3.3) and (2.9). This property is characteristic also for the ellipsoidal bodies in which the strata of equal density are similar to the outer surface [2,3].

6. A special example. It might be instructive to study one explicit, even if unrealistic, solution of eqs. (4.4), (4.5). One such solution is found by assuming that $\omega = \omega(r)$, i.e. that the angular velocity is constant on the equipotential surfaces. Then, after solving eqs. (4.4), (4.5) for $p_{,r}$ and $p_{,\vartheta}$ we obtain the integrability condition $p_{,r\vartheta} = p_{,\vartheta r}$ which is an algebraic equation in $\sin \vartheta$ and $\cos \vartheta$. From that we find

$$f(r)\,\omega^2(r)(r^2 + a^2) = C = \text{const.} \tag{6.1}$$

and, with use of eq. (6.1)

$$2Ga^2 M(r) f(r) = 4Cr(r^2 + a^2) \,, \tag{6.2}$$

$$p = - (C/2a^2) \ln[(r^2 + a^2)(r^2 + a^2 \cos^2 \vartheta)] + C_1 \,, \tag{6.3}$$

where C_1 = const. Now, taking into account eq. (2.9) we have $f = (4\pi)^{-1} \, \mathrm{d}M/\mathrm{d}r$, with $M(0) = 0$. Hence from eq. (6.2)

$$M(r) = [(4\pi C/Ga^2)(r^2 + a^2)^2 - 4\pi Ca^2/G]1/2 \,, \tag{6.4}$$

$$f(r) = [2C/Ga^2 M(r)] r(r^2 + a^2) \,, \quad f(0) = (C/2\pi G)^{1/2} \,, \tag{6.5}$$

and from eq. (6.1)

$$\omega = [a/(r^2 + a^2)] [GM(r)/r]^{1/2} \,. \tag{6.6}$$

The solution is unrealistic in the sense that it cannot obey the boundary condition $p|_{r=r_0} = 0$, and so, as a stellar model, it could only describe the fluid body of a star, to be enveloped into a gaseous atmosphere with a varying pressure at the bottom. However, the solution reveals some interesting features. First, it exhibits the ring-like singularity. Second, in spite of the singularity in ρ the angular velocity field ω is well behaved, as $\omega(0) = (2\pi GC)^{1/4}/a$. The pressure is well behaved everywhere except at the ring where it is infinite. Moreover, the behavior of pressure limits, in the familiar sense, the size of the object: with a suitable choice of $C > 0$ and $C_1 > 0$ the pressure has a definite value for $r = 0 \leqslant \vartheta < \pi/2$, and for each ϑ is a monotonously falling function of r which becomes inescapably negative for r larger than a certain value.

Appendix. The formal solution of eq. (4.5).
Let us introduce the new variables $u = \ln[(r^2 + a^2)^{1/2} \sin \vartheta]$, $v = \ln[(r^2 + a^2)^{1/2}/\sin \vartheta]$. Then we obtain:

$$p_{,v} = - 2\pi Gf(U)M(U)/U(e^{u+v} - a^2 e^{u-v})$$

where $U := (e^{u+v} - a^2)^{1/2}$. The formula for p follows by taking the integral with respect to v of the right-hand side of $p_{,v}$ which consists of functions that are, in principle, known.

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