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A NEWTONIAN MODEL OF THE KERR GRAVITATIONAL FIELD

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In my earlier work [1] I have argued that the gravitational field described by the Kerr metric should correspond to a newtonian field whose equipotential surfaces are confocal ellipsoids. The argument was the following: in the Kerr solution there exists a timelike congruence of curves such that each 3-space locally orthogonal to a curve is foliated into ellipsoids. The 3-space is itself curved, but the curvature is determined only by the mass of the source, and for zero mass the 3-space, along with the spacetime, becomes flat. It is then the simple Euclidean space represented in the spheroidal coordinates.

This argument shows that the Kerr spacetime is composed of confocal ellipsoids of revolution in a way much similar to the way in which the Schwarzschild spacetime is composed of spheres. The ellipsoids are suggestive of equipotential surfaces of the gravitational field. If such connection really exists, then it would be useful to investigate the source of the Newtonian gravitational field whose equipotential surfaces are confocal oblate ellipsoids of revolution.

The present work shows how the desired Newtonian solution can be found. In fact, one source of the field in question is known. It is a shell of finite surface density of matter, the so called homoeoid, coinciding with one of the equipotential ellipsoids ([2], the solution was found by Chasles in 1840). The new solution found here describes an extended perfect fluid ellipsoid endowed with spatial distributions of mass-density, pressure and angular velocity of rotation. It is one of the few solutions of Newtonian hydrodynamics which describe rotating fluid bodies of nonuniform density, and so might be valuable irrespectively of its connection with general relativity.

Let us take the axially symmetric homoeoid of mass M. The potential in the exterior point of cartesian coordinates (X,Y,Z) is:

$$V_H(X,Y,Z) = \int_{\mathcal{V}} \left[(X-x)^2+(Y-y)^2+(Z-z)^2\right]^{-1/2} \times \frac{GM}{2\pi r_o(r_o^2+a^2)}\,\delta\left(\frac{x^2+y^2}{r_o^2+a^2}+\frac{z^2}{r_o^2}-1\right) d_3x \tag{1}$$

where (x,y,z) are coordinates of the current point of integration, r_o is the minor semi-axis of the homoeoid, a is its eccentricity, and $\delta(\cdot)$ is the Dirac delta function centered on the homoeoid. $\mathcal{V}$ is any volume totally containing the homoeoid, and the delta-function is multiplied by constant factors so that the integral yields the correct value of the mass parameter.

Let us change to the spheroidal coordinates:

$$\begin{aligned} x &= \sqrt{r^2+a^2}\,\sin\vartheta\cos\varphi \\ y &= \sqrt{r^2+a^2}\,\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned} \tag{2}$$

and similarly for $(X,Y,Z) \rightarrow (R,\Theta,\Phi)$. The equation $r = \text{const}$ describes an ellipsoid of revolution, different ellipsoids being confocal. Then (1) changes to:

$$V_H(R,\Theta,\Phi) = -\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \int_{f_1(\vartheta,\varphi)}^{f_2(\vartheta,\varphi)} dr\,(r^2+a^2\cos^2\vartheta) \times \tag{3}$$

$$\sin\vartheta\, d^{-1} \frac{GM}{2\pi r_o(r_o^2+a^2)}\,\delta\left(\frac{(r_o^2+a^2\cos^2\;)(r^2-r_o^2)}{r_o^2(r_o^2+a^2)}\right)$$

where $f_1(\vartheta,\varphi) < r_o < f_2(\vartheta,\varphi)$ for all ϑ and φ, f_1 and f_2 being otherwise arbitrary, and:

$$\begin{aligned} d = \Big[&R^2+r^2-2\sqrt{R^2+a^2}\sqrt{r^2+a^2}\sin\Theta\sin\vartheta\cos(\varphi-\Phi) \\ &-2Rr\cos\Theta\cos\vartheta+a^2\sin^2\Theta+a^2\sin^2\vartheta\Big]^{1/2} \end{aligned} \tag{4}$$

Performing the integration with respect to r in (3) we obtain:

$$V_H(R,\Theta,\Phi) = -\frac{GM}{4\pi}\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\, \sin\vartheta/d\,\Big|_{r=r_o} \tag{5}$$

This potential is actually a function of R, its dependence on Θ and Φ being spurious [2-3]. Knowing this, it is easy to guess a continuous density distribution $\varrho(r,\vartheta)$ to replace the δ-like distribution in (3) in such a way that the resulting field depends still only on R. For a body bounded by the ellipsoid $r = r_o$ the required distriburion is:

$$\varrho(r,\vartheta) = f(r)/(r^2+a^2\cos^2\vartheta) \tag{6}$$

where f is an arbitrary function of r. The exterior field of this body is given by:

$$V_{ext}(R) = -G\int_0^{r_o} f(r)\,dr \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\, \sin\vartheta/d \tag{7}$$

The integrand to be integrated over ϑ and φ is here the same as in (5), and so the result of integration is, as before, independent of θ and Φ. If so, then we may substitute $\theta = 0$ in (7) to find:

$$V_{ext}(R) = -\frac{GM}{a}\arctan\frac{a}{R} \tag{8}$$

where:

$$M = 4\pi\int_0^{r_o} f(r)dr \tag{9}$$

Eq. (8) is the unique solution of the Laplace equation for a potential dependent solely on R in the spheroidal coordinates and obeying the appropriate boundary conditions.

Let us now try to solve the Poisson equation:

$$\Delta V = 4\pi G\varrho(x,y,z) \tag{10}$$

assuming that the potential is constant on the ellipsoids confocal with the outer surface of our source. If $V = V(r)$ in the coordinates given by (2), then (10) assumes the form:

$$\left[(r^2+a^2)\frac{d^2V}{dr^2} + 2r\frac{dV}{dr}\right]/(r^2+a^2\cos^2\vartheta) = 4\pi G\varrho(r,\vartheta,\varphi) \tag{11}$$

It is seen from (11) that $\varrho = \varrho(r,\vartheta)$ and that (11) may have a solution only when ϱ is of the form (6).

Solving (11) with ϱ given by (6) we obtain:

$$V_{int}(r) = \int_0^r \frac{4\pi G}{r'^2+a^2}dr'\int_0^{r'} f(r'')dr'' + V(0) \tag{12}$$

where we have assumed $\frac{dV}{dr}(0) = 0$. The value of $V(0)$ should be chosen so that $V_{ext}(r_0) = V_{int}(r_o)$.

We assume now that the oblateness of the body is caused by rotation around the minor axis of the ellipsoid, with the angular velocity field $\omega(r,\vartheta)$. Thus, in the (r,ϑ,φ) coordinates the only nonzero component of velocity is:

$$\frac{d\varphi}{dt} = \omega(r,\vartheta) \tag{13}$$

We find then that the equation of continuity is fulfilled identically, while from the Euler equations of motion we obtain:

$$\omega^2 = \frac{r}{f(r)}p_{,r} + \frac{\cos\vartheta}{f(r)\sin\vartheta}p_{,\vartheta} + \frac{GM(r)}{(r^2+a^2)(r^2+a^2\cos^2\vartheta)} \tag{14}$$

$$\frac{r^2+a^2}{f(r)}\cos\vartheta p_{,r} - \frac{r\sin\vartheta}{f(r)}p_{,\vartheta} + \frac{GM(r)\cos\vartheta}{(r^2+a^2\cos^2\vartheta)} \tag{15}$$

where $M(r)$ is given by (9) with $r_o \rightarrow r$.

We see that (15) is a linear equation for p, solvable with any $f(r)$ and any boundary condition at $r=r_o$. After p is found from (15), ω is determined algebraically by (14).

Let us note that $p=p(r)$ is not a solution of (15). This, together with (6), shows that neither ϱ nor p can be constant on all the surfaces r = const.

If we assume that $\varrho_{,r}<0$, then the surfaces of constant ϱ are more oblate than the surfaces of constant V. Consequently, they cross the outer surface of the body along the circles $\{r=r_o, \vartheta=\text{const}\}$. Note however that if $\varrho_{,r}<0$ all the way from the surface to r=0, then at $\{r=0, \vartheta=\pi/2\}$ ϱ grows to infinity, i.e. a ring-like singularity on the focal ring of the ellipsoids appears. To prevent the singularity, $f(r)$ would have to tend to zero at $r\rightarrow 0$ at least as rapidly as r^2. In this case, however, although ϱ is regular on the ring $\{r=0, \vartheta=\pi/2\}$, it vanishes in the interior of the disk spanned by this ring. This kind of singularity is less ridiculous, but then ϱ cannot be monotonic with respect to r, unless it grows all the way from r=0 to the surface. Consequently, if there is no ring singularity, then ϱ attains its maximum somewhere between the center and the surface or right at the surface.

The ring-like singularity is familiar from the investigations of the Kerr metric source [3-5]: it appeared inescapably whenever one tried to assume that the source is the disk to which the ellipsoids shrink in the limit $r\rightarrow 0$.

It is interesting to note that, in spite of all the aforementioned wild curiosities, in the limit $a\rightarrow 0$ our solution reproduces <u>all</u> the possible spherically symmetric configurations, and not just some of them. Moreover, is shares some properties with spherically symmetric configurations, like e.g. the exterior field, in a given point of space, at fixed a, being determined solely by the mass parameter, M, and not depending on the size of the source. Also, in such a configuration the gravitational force at an interior point P, exerted by the shell lying outside the ellipsoid E passing through P and confocal to the outer surface, is zero. The gravitational force at P is determined solely by the interior of E. This can be seen from (12) and (9).

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