Reprinted from ANNALS OF PHYSICS

Vol. 112, No. 1, May 1978
Printed in Belgium

Ellipsoidal Space-Times, Sources for the Kerr Metric

ANDRZEJ KRASIŃSKI

Institute of Astronomy, Polish Academy of Sciences, Aleje Ujazdowskie 4, 00478 Warszawa, Poland

Received August 6, 1976

The paper develops a systematic derivation of the Kerr metric and its possible sources in a clear geometric manner. It starts with a concise account of previous attempts at constructing an interior Kerr solution. Then a treatment of stationary-axisymmetric space-times, specially fitted to the needs of the following analysis, is presented. A new notion of an ellipsoidal space-time is introduced: it is a space-time in which local rest 3-spaces of some observers split naturally into congruences of concentric and coaxial ellipsoids. It is shown that these 3-spaces are natural spaces to consider the ellipsoidal figures of equilibrium. The investigation is carried out in detail for axially symmetric oblate confocal ellipsoids, but possible generalizations are indicated. The Kerr metric is found to be an ellipsoidal space-time of this special kind. Some remarks concerning an (unfound) explicit interior Kerr solution conclude the paper.

1. INTRODUCTION: A SHORT HISTORY OF THE KERR SOURCE PROBLEM

The Kerr metric [1] which resulted from rather formal investigations of Kerr and Schild (see [2] for a retrospective account) was immediately recognized as describing a space-time exterior to a finite rotating body, thus being very interesting for physical reasons. As the only solution of this kind it has gained much popularity and has given rise to many important works providing a deep insight into mathematical and physical contents of the general relativity theory. The number of papers dealing with various aspects of the Kerr metric is now of the order of 10^2, and is constantly growing.

One of the problems evoked by the Kerr solution, stated already in [1], was the finding of its source, i.e., a metric obeying the Einstein field equations with matter, and joined smoothly to the Kerr metric across some three-dimensional world tube with closed spacelike sections. Curiously, in contrast to considerable progress, e.g., in Kerr black hole physics, the problem of the source remained unsolved until now, in spite of many attempts and partial results. The literature of this subject is rather extended, and it might be useful to summarize briefly the achievements in this field. The following account is hopefully complete until the end of 1975.[1]

Four main lines of attack may be distinguished:

(I) Identifying possible sources by investigation of singularities and physical interpretation of parameters of the Kerr metric.

[1] The author will be grateful for information about any relevant paper not included in the survey.

0003-4916/78/1121-0022$05.00/0

(II) Eliminating some types of sources due to contradictions or inconsistencies to which they lead.

(III) Constructing physically acceptable configurations matched to the Kerr metric only approximately.

(IV) Constructing odd configurations with unphysical properties matched to the Kerr metric exactly.

Group I started with the original paper by Kerr [1] where, by investigating the solution with the use of the Einstein–Infeld–Hoffman method of approximation, it was found that the parameters m and a correspond, respectively, to mass and angular momentum per unit mass of the source. The identification of a was later confirmed by various methods by Boyer and Price [3] (comparison with the Lense–Thirring [4] form of the field of a rotating liquid sphere), Cohen [5] (use of the definition of angular momentum through integrals of the energy-momentum tensor contracted with appropriate Killing vectors), Léauté [6] (the same method as in [3]), Moss and Davis [7] (use of Komar's [8] formulation of conservation laws). These papers provided a preliminary picture of the source, though a very vague one ("something that rotates" was the message).

Janis and Newman [9] devised a method for analyzing gravitational multipole moments in the exact general-relativistic formalism, and investigated the moments of the Kerr field [10]. From their formal similarity to electromagnetic moments of a rotating ring of charge it was concluded that the source of the Kerr field is a rotating ring of uniformly distributed mass m having the radius a, i.e., it is placed right in the ring singularity of the metric. In the same paper [10] a remark due to Kerr [11] is mentioned, that it is rather the disk spanned by this ring that should be interpreted as a source because on it the metric coefficients suffer a nonuniqueness of a branch-point type.

Recently Léauté [12] used Bel's method [13] of identifying singular points of various stationary metrics, and found that the source of the Kerr metric may be described as a point particle of mass m equipped with an additional spin degree of freedom. This result, however, has no direct physical interpretation, as Léauté's analysis was carried on in a conformal transform of the rest space of observers having the timelike Killing field as their four-velocity. The conformal transformation involved is singular on the surface of the ergosphere, and its image is just the singular point interpreted as a point particle.

In group I we may also mention Misra [14], who obtained an interesting result that every *static* axially symmetric (nonspherical) empty space-time must possess a ring singularity. However, it was not verified if this singularity is really irremovable.

The first paper of group II was that of Boyer [15], where it was shown that a rigidly rotating perfect fluid source must be bounded by a surface obeying a definite algebraic equation. Later the same author [16], tried to obtain a similar equation for a nonrigidly rotating perfect fluid source. However, since there is an infinite variety of possible nonrigid rotations as opposed to the unique rigid one, the result contains such a high degree of arbitrariness that its practical usefulness is very small.

Hernandez [17, 18] eliminated a very special combination of assumptions about the source. He divided the Kerr metric into two parts: the spherically symmetric (Schwarzschild) part and "the rest." Then he assumed that the same division may be made for the interior metric searched for and that the spherical interior part is just the interior Schwarzschild solution of uniform density. Finally, he strove to match separately the spherical parts to each other, and "the rests" to each other, assuming a very special form of the interior "rest," and using the perfect fluid field equations. It appeared that this program cannot be performed. This failure will be simply explained at the end of the present paper. Hernandez's suggestion that the Kerr metric has no fluid source at all is based merely on the result described above, and thus must be treated as unjustified.

Herlt [19] investigated systematically the consequences of Boyer's equation [15] and apparently showed that a rigidly rotating perfect fluid source cannot be matched to the Kerr metric if the configuration is to have a meaningful limit $a \to 0$, and its pressure is to be positive everywhere. This statement was shown to be false by Roos [20], who pointed out a few mistakes in Herlt's analysis. Roos investigated the problem of integrability of the Einstein field equations with a perfect fluid or anisotropic nondissipating fluid as a source, and with the boundary condition that their solutions are matched to the Kerr metric. It appeared that, with some limitations on the shape of the bounding hypersurface, these equations form an integrable set, at least in a finite neighborhood of that hypersurface, whether the fluid rotates rigidly or not. Based on this fact Abramowicz et. al. [21] made another investigation of Boyer's equation [15] and obtained a few inequalities to be fulfilled by parameters describing the source. These do not exclude the possibility of a fluid source as the authors sorrowfully admit. Their paper contains an overt misinterpretation of Thorne's statement [22] that the Kerr metric "cannot represent correctly the external field of *any* realistic stars." This meant: not any arbitrary—because all the multipole moments are exactly determined by the angular momentum parameter a which shows that the source must have a very definite structure. However, the authors of [21] have taken it literally to mean: not any at all.

The analysis of Herlt was done again, with use of the very same method as in [19], by de Felice *et al.* [23]. They found possible shapes of the source, and recovered partially the results which Herlt obtained and discarded on the grounds of his mistakes. Paper [23] is an improved version of a fragment of an earlier paper by de Felice [24], where the same problem was investigated in the weak-field approximation, with less conclusive results.

Finally, Abramowicz *et al.* [25] stated that "pressureless disk cannot be the source for the Kerr metric," this conclusion resulting from the assumption that the two-dimensional surface of the disk must be a closed one, which is plainly false.

The information supplied by group III is not very reliable because matching two solutions is a mathematical procedure requiring definite yes-or-no conclusions. Approximately almost everything can be matched to anything if the approximation is cleverly adjusted. This opinion is confirmed by the variety of "approximate sources" for the Kerr metric.

The first paper in this series is due to Cohen [26] who used the method devised by him and Brill [27]. It is shown that up to linear terms in the angular velocity the Kerr metric can be matched to a rotating spherical shell of zero thickness (so in fact this paper is a hybrid of all the poor features of groups III and IV). Cohen remarks that a full perfect fluid sphere, and possibly many other configurations, could serve as approximate sources in this formalism.

The same configuration as in [26] was investigated up to square terms in angular velocity by de la Cruz and Israel [28]. They confirmed the first-order results of Cohen [26] that the shell is spherical, of uniform surface density of matter, rigidly rotating and having a flat space-time inside. In the second order all these properties are lost unless the shell is placed right on the Schwarzschild sphere. It is emphasized that the convergence of this approximate procedure is not guarenteed, but if it is convergent then there exists an infinite variety of shells of various shapes containing empty (not necessarily flat) spaces inside and Kerr space-times outside.

This last statement was even strengthened by Hartle and Thorne [29], who obtained an empty space metric valid outside any slowly rotating portion of perfect fluid. This agreed, up to square terms in angular velocity, with the Kerr metric.

McCrea [30] used the method of Synge [31] to find the interior and exterior fields of a rigidly rotating material sphere, and showed that up to k^2 (where k is some small parameter) the exterior field is the same as that of Kerr. At the same time the Florides–Synge method [32] was used by Florides [33] to show that up to k^5 the Kerr metric may be matched to an interior solution describing a rotating sphere of nonperfect fluid (pressure anisotropic). Up to k^3 its rotation is rigid; in higher orders it becomes nonrigid.

Arkuszewski *et al.* [34] investigated the linearized version of the Einstein–Cartan field equations, and have shown that in the linear approximation the Kerr metric may be matched to a static sphere of Weyssenhoff fluid (composed of spinning particles). (Speaking about other theories of gravitation let us mention in passing that an exact counterpart of the Kerr metric has been found in the Brans–Dicke theory by McIntosh [35]).

Finally, Florides [36] used the same method as before to match the Kerr metric up to k^5 to a rigidly rotating oblate spheroid of nonperfect fluid (anisotropic pressure).

Now let us consider group IV. It was started by Keres [37], who devised a method for finding Newtonian limits of relativistic gravitational fields, and used it to investigate the Newtonian limit of the Kerr metric. This appeared just the same as the (Newtonian) field of a rotating disk spanned by a ring of radius a. The interior of the disk had negative mass whose density diverged to $-\infty$ when approaching the circumferential ring. The ring, however, contained a compensating infinite positive mass, so that the net mass was just m.

Later, Israel [38] tried to remove the arbitrariness connected with the choice of the boundary of a source by assuming a "minimal" source, i.e., the disk spanned by the singular ring. He used the full general-relativistic theory of Lanczos [39] connecting surface energy–momentum densities to discontinuities in the second fundamental form of the source's boundary. In this way he arrived at precisely the result of

Keres [37]. Though such an artificial source can hardly be taken seriously, the complete agreement between two different methods is really remarkable. Israel concluded that we do not understand the Kerr geometry under the inner horizon, and perhaps a radically new approach is needed.

Still another disk, a "microgeon" made of gravitationally bound electromagnetic field, was matched to the Kerr metric by Burinskii [40]. Gürses and Gursey [41] and Collas and Lawrence [42] found a source made of nonperfect fluid whose pressure was anisotropic with at least one principal value negative. This solution was matched to the Kerr metric across an ellipsoid $r = r_0$ in the Boyer–Lindquist [43] coordinates. It will be mentioned once more at the end of the present paper, in connection with our results.

Hogan [44] obtained a nonperfect fluid source (anisotropic pressure) enveloped in a "crust" of zero thickness, finite surface density of matter, and somewhat arbitrary in shape (a "deformed ellipsoid"). This could perhaps be considered as a primitive model of a planet, but in that case one would prefer the "crust" to be of finite thickness.

Most recently Hamity [45] has found a source in the form of a rigidly rotating disk with regular interior and singular edge. The interior of the disk consists of matter having isotropic tension and zero energy density. The tension diverges to ∞ when approaching the edge. Such behavior resembles the Keres–Israel disk [37–38]. However, Hamity did not cite those authors and made no comparison of the results.

So many unsuccessful efforts aroused a gradually growing pessimism. At the height of the pessimism there appeared an opinion that the Kerr metric might have no other source than a black hole [46]. Whatever arguments may be used to support this opinion, one thing is clear: such a statement was never proved. The lack of success in searching for the source may as well indicate the inadequacy of the methods used, and is in itself not even a suggestion that the source must be something exotic. Therefore I have tried to devise a simpleminded method based solely on such assumptions whose geometrical meaning is clear and easily understood.

This paper presents such an approach. It will prove to be of some real value only if it actually provides a source for the Kerr metric. This goal has not yet been achieved, but it seems that some new possibilities have opened up. At any rate, it is a new approach, and the foregoing introduction was intended to justify the author's undertaking.

Section 2 is a treatment of stationary-axisymmetric space-times, arranged slightly unconventionally for the needs of the succeeding sections. In Section 3 a new structure called ellipsoidal space-time is introduced. It is a space-time which distinguishes geometrically a congruence of concentric and coaxial ellipsoids. The exposition is directed toward the main goal of obtaining a source for the Kerr solution, so only axially symmetric oblate confocal ellipsoids are considered in detail, but a few possible generalizations of the procedure are indicated as problems for future investigation. A general metric form for an ellipsoidal space-time filled with matter is obtained. In Section 4 empty ellipsoidal space-times are considered, and the Kerr metric emerges there as a corollary of a derivation somewhat similar to the standard derivation of spherically symmetric line elements. Other papers which touched the main ideas of the treatment are mentioned. Finally, in Section 5, one more unsuccessful trial of group II is presented together with some general remarks concerning the searched interior solution.

2. Stationary — Axisymmetric Space-Times

In this section we shall mainly appeal to the intuitive geometrical background of the definitions given. We shall only explain what and why we assume, without insisting on the "economy of deduction" (i.e., conclusions as strong as possible from assumptions as weak and as few as possible), and without using the field equations. In fact, nothing will be proved, only the contents of definitions will be slightly transformed. More rigorous treatments of the subject may be found elsewhere (e.g., [22, 47–52])

We realize a stationary space-time as one in which there exists a distinguished family of observers who see the space-time geometry unchanged as their proper time flows. An axisymmetric space-time is one whose geometry does not change under rotations around a given spacelike line. Consider now a space-time which is both stationary and axisymmetric. Let the distinguished observers wait idly for a time t, and then rotate the space-time instantaneously by an angle φ. We feel intuitively that the result (nothing has changed!) will be the same as if we first rotated the space-time by φ, and then let the observers wait for the time t. It means that the two symmetry operations commute. This property is specified by assuming that there exist two independent Killing vector fields $\underset{(\varphi)}{k}{}^{\mu}$ and $\underset{(t)}{k}{}^{\mu}$ which obey the equation:

$$[\underset{(t)}{k}, \underset{(\varphi)}{k}]^{\mu} = \underset{(t)}{k}{}^{\rho} \underset{(\varphi)}{k}{}^{\mu}{}_{,\rho} - \underset{(\varphi)}{k}{}^{\rho} \underset{(t)}{k}{}^{\mu}{}_{,\rho} = 0. \tag{2.1}$$

The vector $\underset{(t)}{k}$ is timelike, being tangent to the world lines of the distinguished observers. It is easy to prove (see, e.g., [53]) that to any two commuting linearly independent vector fields such coordinates may be adjusted in which:

$$\underset{(t)}{k}{}^{\mu} = \delta^{\mu}{}_{0}, \qquad \underset{(\varphi)}{k}{}^{\mu} = \delta^{\mu}{}_{3}. \tag{2.2}$$

With such Killing vectors the metric tensor of the space-time does not depend on x^0 and x^3. It is important to know what transformations of coordinates preserve this property. Let us note then that any linear combination of Klling vectors with constant coefficients is also a Killing vector. Therefore the searched transformations need not preserve each Killing vector individually, but only transform any such linear combination of them into another combination with constant coefficients. It is easy to find that these transformations are given by :

$$\begin{aligned} x^0 &= Ax^{0'} + Bx^{3'} + F^0(x^{1'}, x^{2'}), \\ x^1 &= F^1(x^{1'}, x^{2'}), \\ x^2 &= F^2(x^{1'}, x^{2'}), \\ x^3 &= Cx^{0'} + Dx^{3'} + F^3(x^{1'}, x^{2'}), \end{aligned} \tag{2.3}$$

where $F^\mu(x^{1'}, x^{2'})$, $\mu = 0, 1, 2, 3$, are arbitrary functions of two variables, and A, B, C, D, are arbitrary constants subject to the condition:

$$(AD - BC) \cdot \frac{\partial(F^1, F^2)}{\partial(x^{1'}, x^{2'})} \neq 0. \tag{2.4}$$

In this point our approach differs from the standard ones [47–52]. It is usually assumed that the trajectories of the Killing vector $\underset{(\varphi)}{k}$ are closed (at least in "spatial infinity"), with $\{x^3 = \varphi + 2\pi$ (other coordinates fixed)$\}$ corresponding to the same point as $x^3 = \varphi$. This singles out a particular subclass of the transformations (2.3) in which $B = 0$, $D = 1$; otherwise the period of φ would be other than 2π, and shuffling of periodic φ with nonperiodic t would cause geometrical inconsistencies. We want to retain the freedom to choose B and D arbitrary. Then the $\underset{(\varphi)}{k}$-trajectories will not, in general, be closed: they will be screw- lines on the (t, φ)-cylinders.

Suppose that the lines of $x^{3'}$ in (2.3) are closed, and consider the coordinate x^0. If $B \neq 0$ then increasing $x^{3'}$ from 0 to 2π, and keeping $x^{0'} = \text{const}$, we return to the starting point, but with x^0 increased by $2\pi B$. This means that we have introduced a time coordinate which suffers a jump: it differs by nearly $2\pi B$ for two points of which one has $x^{3'}$ just above 0, and the other just below 2π, $x^{0'}$ being the same for both. It follows that after x^3 is increased by its full period from 0 to $2\pi D$ (keeping x^0 constant) we arrive not back to the starting point, but to a point shifted by $2\pi B$ along the x^0-line given by $x^3 = 0$. Therefore it must be assumed that the x^3-lines are finite segments of screw lines, topologically closed at the $\{x^3 = 0\}$-end, and open at the $\{x^3 = 2\pi D\}$-end. The trajectories of $\underset{(\varphi)}{k}$ are composed smoothly of these segments, only their parametrization behaves wildly. This is a typical difficulty if we use two intersecting congruences of screw lines as coordinates on a cylinder (see Fig. 1).

If we choose $B = 0$ in (2.3) then the x^3-lines are still closed and there is no jump. However, since the metric depends neither on x^0 nor on x^3, we have no straightforward means to check if the (x^0, x^3) we actually use are nonjumping, until we relate them to some well-known geometrical structure. This problem cannot be removed just by the arbitrary procedure of "identifying points;" it is a physical problem to be solved. Consequently, we must admit (2.3) with B arbitrary to be able to remove jumps if necessary.

Let us now consider the motion of the distinguished observers. It may happen that they are at rest with respect to some reference frame defined by (2.2) (it is possible when there exist two mutually orthogonal linear combinations of the vector fields $\underset{(t)}{k}$ and $\underset{(\varphi)}{k}$ with constant coefficients). The space-time is then called static. We shall be interested in the other case when the observers are in motion with respect to *every* reference frame given by (2.2). Such a space-time is called stationary nonstatic.

Owing to the assumed symmetries the observers may move only in the $\varphi = x^3$-direction, otherwise their four-velocity would not be colinear with any combination of the Killing vectors, and their world lines would not coincide with any trajectories of the symmetry group. In a given time interval $\Delta t = \Delta x^0$ each of the observers will pass through a sequence of values of the angle $\varphi = x^3$. If we reverse the positive-time direction, each observer will pass this same sequence of values of φ in a reversed order. Thus the time-reversal yields the same effect as the reversal of the direction of rotation of observers. Consequently, if the coordinate system is adapted so as to describe these two reversals by $t \to -t$ and $\varphi \to -\varphi$, respectively, then these transformations cancel each other, and the simultaneous inversion:

$$(t, \varphi) \to (-t, -\varphi) \tag{2.5}$$

is a symmetry operation. This property implies that every component of the metric tensor which would change its sign under the inversion (2.5) must vanish, i.e.:

$$g_{01} = g_{02} = g_{13} = g_{23} = 0. \tag{2.6}$$

Notice, however, that the inversion of time flow and direction of rotation is described simply by (2.5) only in a subclass of coordinate systems given by (2.3). The inversion of $(x^{0\prime}, x^{3\prime})$ results in a more complicated transformation of (x^0, x^3) if

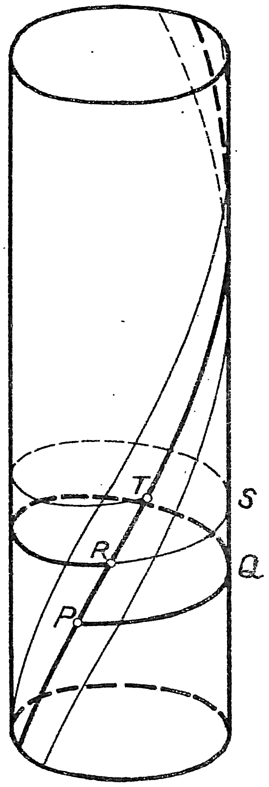

FIG. 1. Skew coordinates on a cylinder. Steeper lines are full screw-lines x^3 = const, parametrized by x^0. Less tilted lines are finite segments of screw lines, on which x^0 = const, parametrized by x^3. The x^0-coordinate suffers a jump $\Delta t = 2\pi B$ when crossing the line PRT (on which $x^3 = 0 \sim 2\pi D$): the segment PQR (R excluded) has $x^0 = 0$, the segment RST (R included) has $x^0 = 2\pi B$. The metric form alone does not allow one to distinguish such "jumping" coordinates from regular cylindrical coordinates.

$F^0 \neq 0 \neq F^3$. If we want to describe the considered inversion by (2.5), possibly combined with symmetry transformations generated by $\underset{(t)}{k}$ and $\underset{(\varphi)}{k}$, then we may admit (2.3) only with $F^0, F^3 = \text{const}$.

We have thus shown that the metric of a stationary-axisymmetric space-time may be put in the form:

$$ds^2 = g_{00}dt^2 + 2g_{03}dt\,d\varphi + g_{33}d\varphi^2 + g_{AB}dx^A dx^B \tag{2.7}$$

where $A, B = 1, 2$, and all the components of the metric depend only on x^1 and x^2.

The terms gathered in $g_{AB}dx^A dx^B$ form a two-dimensional metric, covariant under the transformations (2.3) with $F^0, F^3 = \text{const}$. It is known (see, e.g., [54]) that a two-dimensional metric form may always be represented as explicitly conformally flat,

$$g_{AB}dx^A dx^B = \Phi^{-2}(x^1, x^2)[(dx^1)^2 + (dx^2)^2].$$

We will content ourselves with a less special coordinate system in which the subtensor g_{AB} is only diagonal, so:

$$g_{AB}dx^A dx^B = g_{11}(dx^1)^2 + g_{22}(dx^2)^2. \tag{2.8}$$

This coordinate system is specified exactly to the transformations (2.3) with F^0, $F^3 = \text{const}$ and (F^1, F^2) obeying:

$$g_{11}F^1{}_{,1'}F^1{}_{,2'} + g_{22}F^2{}_{,1'}F^2{}_{,2'} = 0. \tag{2.9}$$

Suppose now that in the space-time considered there is matter present, and the distinguished observers move along the flow lines of matter, which are thus trajectories of the symmetry group. Then the four-velocity u^α fulfills the equations:

$$u^1 = u^2 = 0 \tag{2.10}$$

and $[u, \underset{(t)}{k}] = [u, \underset{(\varphi)}{k}] = 0$, which means:

$$u^\alpha = U(x^1, x^2)\,\delta^\alpha{}_0 + V(x^1, x^2)\,\delta^\alpha{}_3 \tag{2.11}$$

where U and V are arbitrary functions of two variables.

Since the Kerr solution itself is stationary and axially symmetric, it will be reasonable to look for its source having the same symmetry group. However, it is possible to start with metrics of an even narrower class than that given by (2.7), (2.8). They are connected with ellipsoidal space-times considered below.

3. Ellipsoidal Space-Ttimes

In Newtonian hydrodynamics one is able to gain any exact information about a finite rotating portion of fluid only in case it has the shape of an ellipsoid [55, 56].

Therefore, it will be natural to start analogous investigations in general relativity considering the same simplest case. It will be most convenient to use the ellipsoids as defining a coordinate system in the space-time. To do this, however, we must first define what an ellipsoid *is* in curved spacetime, i.e., at least find the metric form of an ellipsoidal surface. For simplicity, and in order to keep inside the class of axially symmetric space-times, we shall consider ellipsoids of revolution only, thus leaving behind us:

Problem I. Make an analysis, analogous to that given below, for nonsymmetric ellipsoids.

Imagine now the three-dimensional Euclidean space filled with a congruence of ellipsoids of revolution with common center and common axis of symmetry. Let the two semiaxes of each ellipsoid be quite independent, i.e., do not assume any correlation between the shapes of different ellipsoids. Let the semiaxis lying along the symmetry axis be taken as the coordinate r in space, so that $r = \text{const}$ on a fixed ellipsoid. Let Φ be the azimuthal angle measured around the symmetry axis, and ϑ be a third coordinate defined at will. It is most convenient to define (r, ϑ, Φ) in terms of Cartesian coordinates (x, y, z) as follows:

$$\begin{aligned} x &= g(r) \sin\vartheta \cos\Phi, \\ y &= g(r) \sin\vartheta \sin\Phi, \\ z &= r\cos\vartheta, \end{aligned} \tag{3.1}$$

where $g(r)$ is an arbitrary function whose value equals the other semiaxis of the $r = \text{const}$ ellipsoid. In the (r, ϑ, Φ)-coordinates the equation of an ellipsoid $[(x^2 + y^2)/g^2(r)] + (z^2/r^2) = 1$ is an identity so each ellipsoid is described simply by $r = \text{const}$. The metric of the Euclidean space now assumes the form:

$$\begin{aligned} dx^2 + dy^2 + dz^2 = {}& (g_r{}^2 \sin^2\vartheta + \cos^2\vartheta)\, dr^2 + 2(gg_r - r)\sin\vartheta\cos\vartheta\, dr\, d\vartheta \\ & + (g^2\cos^2\vartheta + r^2\sin^2\vartheta)\, d\vartheta^2 + g^2\sin^2\vartheta\, d\Phi^2. \end{aligned} \tag{3.2}$$

Note that the new coordinates are orthogonal only when $gg_r - r = 0$, i.e., $g^2 - r^2 = \text{const}$, which means that the ellipsoids are confocal, prolate for $g^2 - r^2 < 0$, and oblate for $g^2 - r^2 > 0$ (for $g^2 = r^2$ they reduce to spheres). For our purposes it will be enough to consider the case of confocal oblate ellipsoids, so we assume now:

$$g^2 - r^2 = a^2 \tag{3.3}$$

and consequently we leave for future investigation:

Problem II. Consider the cases of ellipsoids correlated in other ways, e.g., geometrically similar ones for which $g/r = \text{const}$.

With (3.3) the metric (3.2) reduces to:

$$dx^2 + dy^2 + dz^2 = [(r^2 + a^2 \cos^2 \vartheta)/(r^2 + a^2)]\, dr^2 + (r^2 + a^2 \cos^2 \vartheta)\, d\vartheta^2 + (r^2 + a^2) \sin^2 \vartheta\, d\Phi^2. \tag{3.4}$$

We obtain from here the metric form of the surface of an ellipsoid of revolution simply putting $r = \text{const}$. For one ellipsoid the assumption (3.3) is not restrictive because the geometry of one surface does not depend on the properties of the congruence in which is is imbedded.

Coordinate systems of the kind (3.1) were used by Misra [14], Zipoy [57], Morgan and Morgan [58] and Slivinskii [59] to analyze some static axisymmetric solutions. There, however, no geometrical structures in space-time were connected with the ellipsoids. The flat space metric precisely in the form (3.4) was obtained by Koppel [60] as a "nonrelativistic limit" of the Kerr solution.

We need now a metric form of a three-dimensional, axisymmetric *curved* space in which the (r, ϑ, Φ)-coordinates are orthogonal, and the surfaces $r = \text{const}$ are ellipsoids of revolution, the same as resulting from (3.4). It is easily seen to be:

$$ds^2{}_3 = f^2(r, \vartheta)\, dr^2 + (r^2 + a^2 \cos^2 \vartheta)\, d\vartheta^2 + (r^2 + a^2) \sin^2 \vartheta\, d\Phi^2 \tag{3.5}$$

where the dependence of f on r and ϑ should be found from geometrical considerations. This is:

PROBLEM III. How does the function $f(r, \vartheta)$ depend on the shape of ellipsoids forming the congruence?

Before proceeding to four-dimensional space-time let us recall that our ellipsoids should be connected with surfaces of rotating fluid bodies. In Newtonian mechanics the statement that a rotating body has ellipsoidal shape is of universal meaning, irrespective of the observer's motion. In general relativity theory the shape of a surface will depend on the observer performing its description, owing to deformations resulting from relative motions. Consequently, the surface of a rotating body may look like an ellipsoid for a limited class of observers only, and we must decide for which one it does.

Notice then that for an observer comoving with the body the problem of determining its surface's shape, i.e., of determining the direction normal to the surface in a small neighborhood, is a problem of Newtonian statics. He merely has to find the vectorial sum of two constant forces: the gravity force and the centrifugal force. If the observer resides inside the body and wants to determine the shape of equipressure surfaces, then a third force, the hydrostatic pressure of layers above him, appears, but the problem is still that of statics. Now, statics pursued in general relativistic language and in the rest frame of the static system is just identical to Newtonian statics. We can therefore reasonably expect that, in the local rest frame of an observer comoving with matter, what was an ellipsoid in Newtonian theory will stay an ellipsoid in general relativity. The natural 3-space to consider the ellipsoidal figures of equilibrium in general relativity is therefore the union of local rest spaces of the observers comoving with matter, and (3.5) is the metric form of that 3-space. The metric 3-tensor $h_{\alpha\beta}$ defined by (3.5) should be identified with:

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta \tag{3.6}$$

where $g_{\alpha\beta}$ is the searched space-time metric, and u_α is the four-velocity field of matter. We now define:

DEFINITION I. A space-time filled with matter is called ellipsoidal if the metric of the local rest spaces of matter is given by (3.5). (This definition should be appropriately modified if we want to relax the special assumptions about ellipsoids which led to (3.4) and consider some of the problems.)

Having in mind the sources for the Kerr metric we shall deal only with those ellipsoidal space-times which are at the same time stationary and axisymmetric. However, here arises:

PROBLEM IV. What is the metric form of the most general ellipsoidal space-time?

With use of (2.7), (2.8), (2.11), and (3.6) we find:

$$\begin{aligned} -h_{\alpha\beta}dx^\alpha dx^\beta = {} & g_{11}dr^2 + g_{22}d\vartheta^2 \\ & + (g_{00}g_{33} - g^2{}_{03})\, U^2[d\varphi - (V/U)\, dt]^2 \end{aligned} \tag{3.7}$$

and this should be identified with (3.5). We have therefore:

$$(g_{00}g_{33} - g^2{}_{03})\, U^2 = (r^2 + a^2)\sin^2\vartheta, \tag{3.8}$$

$$g_{11} = f^2(r, \vartheta); \tag{3.9}$$

$$g_{22} = r^2 + a^2\cos^2\vartheta; \tag{3.10}$$

$$d\Phi = d\varphi - (V/U)\, dt; \tag{3.11}$$

and, since $g_{\alpha\beta}u^\alpha u^\beta = 1$ must hold:

$$g_{00}U^2 + 2g_{03}UV + g_{33}V^2 = 1. \tag{3.12}$$

The transition $g_{\alpha\beta} \to h_{\alpha\beta}$ is a projection, so the inverse transition $h_{\alpha\beta} \to g_{\alpha\beta}$ is necessarily nonunique, and must involve an arbitrary function. Let this function be g_{33}, and let us introduce for brevity:

$$k^2 \stackrel{\text{def}}{=} g_{33} + (r^2 + a^2)\sin^2\vartheta. \tag{3.13}$$

Then, after the equations (3.8) and (3.12) are solved for $g_{\alpha\beta}$, we obtain:

$$\begin{aligned} ds^2 = {} & [U^{-1}(1 - kV)\, dt + k\, d\varphi]^2 \\ & - f^2\, dr^2 - (r^2 + a^2 \cos^2 \vartheta)\, d\vartheta^2 \\ & - (r^2 + a^2) \sin^2 \vartheta [d\varphi - (V/U)\, dt]^2 \end{aligned} \tag{3.14}$$

where U, V, k, f are unknown functions of r and ϑ to be determined from the field equations. Note that two of the four functions are velocity components of matter defined by (2.11).

4. Empty Ellipsoidal Space-Times

In the above derivation it was necessary to have matter in the space-time to define the 3-spaces splitting into ellipsoids. Now it is easy to generalize the definition of an ellipsoidal space-time to the vacuum-case:

Definition II. An empty space-time is called ellipsoidal if there exists in it any congruence of observers whose local rest spaces have the metric given by (3.5).

(The remark following Definition I is still valid.)

Both Definitions I and II give rise to:

Problem V. What is the operational method to recognize an ellipsoidal space-time?

The definitions suggest a straightforward way: If there is matter, find the metric of its local rest spaces, and try to put it in the form (3.5). If there is no matter, guess first which observers should have local rest spaces described by (3.5). This method is effective if we are able to prove that some space-time *is* ellipsoidal. The essence of Problem V is: how to prove unequivocally that a space-time is not ellipsoidal, when it is not? How to get rid of the possibility that it is ellipsoidal, but we are not clever enough to show it? One would like to have some technical criterion, perhaps analogous to Killing equations which are unique indicators of symmetries.

Here the Kerr solution [1] enters into our considerations. It is immediately recognized as describing an empty ellipsoidal space-time. The same is true for the "Kerr solution with electric charge" found by Newman *et al.* [61], and we shall consider this slightly more general case. Let us perform on the metric given in [61] the following coordinate transformation:

$$\begin{aligned} & u = t + a\varphi' - \int [(r^2 + a^2)/D]\, dr, \\ & r = r', \qquad \vartheta = \vartheta', \\ & \varphi = \varphi' - \int (a/D)\, dr, \\ & D \stackrel{\text{def}}{=} r^2 - 2mr + e^2 + a^2. \end{aligned} \tag{4.1}$$

The result, on dropping primes, is:

$$ds^2 = \left(1 - \frac{2mr - e^2}{r^2 + a^2 \cos^2 \vartheta}\right) dt^2 + 2a \left[1 - \frac{(2mr - e^2)\cos^2 \vartheta}{r^2 + a^2 \cos^2 \vartheta}\right] dt\, d\varphi$$
$$+ \left[a^2 - \frac{(2mr - e^2)\, a^2 \cos^4 \vartheta}{r^2 + a^2 \cos^2 \vartheta} - (r^2 + a^2)\sin^2 \vartheta\right] d\varphi^2$$
$$- \frac{r^2 + a^2 \cos^2 \vartheta}{r^2 - 2mr + e^2 + a^2}\, dr^2 - (r^2 + a^2 \cos^2 \vartheta)\, d\vartheta^2. \tag{4.2}$$

Comparison with (3.14) and (3.5) shows that this is an ellipsoidal space-time, with the four-velocity of the observers referred to in Definition II given by:

$$\begin{aligned} U &= [(r^2 + a^2)/D]^{1/2}; \\ V &= g_{33}^{-1}[-U g_{03} \pm a\Delta^{1/2}]; \\ \Delta &= 1 - (2mr - e^2)\cos^4 \vartheta/(r^2 + a^2 \cos^2 \vartheta) \end{aligned} \tag{4.3}$$

where g_{03} and g_{33} should be read off from (4.2). The double sign in V means that, in the region $\Delta > 0$, there are two congruences of observers fulfilling the conditions of Definition II. The region $\Delta > 0$ contains the whole outside of the ergosphere, $g_{00} > 0$. We shall not deal here with the regions inside the ergosphere and under the horizon as they are supposed to be cut off from the space-time to make the place for a source.

The proper Kerr metric results from (4.2) and (4.3) in the case $e = 0$. It is then represented in the coordinate system related to that of Boyer and Lindquist [43] $(t_{BL}, \varphi_{BL}, r_{BL}, \vartheta_{BL})$ by the simple transformation:

$$\begin{aligned} &t_{BL} = -t - a\varphi; \\ &r_{BL} = r, \qquad \vartheta_{BL} = \vartheta, \qquad \varphi_{BL} = \varphi. \end{aligned} \tag{4.4}$$

This transformation shows that either t or t_{BL} is a "jumping" time. The clear geometrical interpretation of t_{BL} suggests that it is rather t which jumps.

It is not clear what Newtonian structures correspond to the ellipsoids defined by the Kerr solution. The most obvious suggestion is that the ellipsoids are relativistic analogs of equipotential surfaces. The only known Newtonian source having confocal ellipsoids of revolution as equipotential surfaces is an infinitesimally thin homeoid [56]. This is not a satisfactory source; one would like to "fill in" the interior of the homeoid. This would be a problem both in Newtonian theory and in general relativity: how to fill in the homeoid with some nonexotic matter (most satisfactorily a perfect fluid) so that the structure of equipotential surfaces is preserved. There is also another problem: the surface of a rotating body does not coincide with any surface of constant gravitational potential. It is determined by the *effective* potential (gravitational plus centrifugal), and is usually more oblate. So if we interpret the ellipsoids in the Kerr metric as equipotential surfaces then we should not expect the surface of the source to be one of these ellipsoids. This suggestion is supported by the fact proven by

Roos [71] that a perfect-fluid source rotating in an arbitrary way cannot be matched to the Kerr metric across the hypersurface $r_{BL} =$ const. It seems that here is the central point of the problem.

It was already recognized by Misra [62], Tiomno [63], Newman and Winicour [64], Florides [36] and Hogan [65, 66] that "the Kerr solution has something to do with the external field of a rotating homeoid" (i.e., a body of spheroidal shape), Misra's contribution being merely the cited sentence. Tiomno [63] has shown that the Kerr–Newman [61] electromagnetic field is identical to that of a rotating charged oblate ellipsoid of revolution in flat space, with infinite conductivity, and magnetic susceptibility also infinite or equal to that of vacuum. In Hogan's papers [65, 66] the spheroids are places where the Kerr–Schild [2] null geodesics originate. It is suggested there that the spheroids do not rotate. Newman and Winicour [64] have disdained geometrical intuition, and hidden it behind formal computations. They have found that the Kerr–Schild null congruence originates on the surfaces of confocal rotating oblate spheroids, but they rejected the connection between these and the Maclaurin spheroids. They recognized that their ellipsoids rotate with *different* angular velocities, but they used Boyer's condition for *rigid* rotation [15] to conclude (without clear arguments) that the Boyer boundary may coincide with one of the ellipsoids only when the latter is on the Killing horizon r_+. Finally, we are in agreement with the approximate result of Florides [36] who has shown that, up to fifth order in the Florides-Synge method [32], the Kerr metric represents the exterior field of a rotating spheroidal body. The focuses of his ellipsoids are $2 \times 5^{1/2}$ a part.

In the end, let us illustrate the difficulties connected with Problem V by an example. The Kerr metric has been generalized by Demianski [67] to include the cosmological constant:

$$
\begin{aligned}
ds^2 = {} & \left[1 - \frac{2mr}{r^2 + a^2\cos^2\vartheta} + \frac{1}{3}\Lambda(r^2 + a^2\cos^2\vartheta)\right] dt^2 \\
& + 2a\sin^2\vartheta\left[\frac{2mr}{r^2 + a^2\cos^2\vartheta} - \frac{1}{3}\Lambda(r^2 + a^2)\right] dt\, d\varphi \\
& - \sin^2\vartheta\left[\frac{2mra^2\sin^2\vartheta}{r^2 + a^2\cos^2\vartheta} + (r^2 + a^2)\left(1 - \frac{1}{3}\Lambda a^2\right)\right] d\varphi^2 \\
& - \frac{r^2 + a^2\cos^2\vartheta}{\frac{1}{3}\Lambda(r^2 + a^2)\, r^2 + r^2 - 2mr + a^2}\, dr^2 \\
& - \frac{r^2 + a^2\cos^2\vartheta}{1 - \frac{1}{3}\Lambda a^2\cos^2\vartheta}\, d\vartheta^2 .
\end{aligned}
\tag{4.5}
$$

(The same metric, in a different coordinate system, was independently obtained by Frolov [68]. Another independent derivation due to Carter [69] gave a result discrepant with (4.5), which indicates a computational error, most probably in [69].) Perhaps this metric is ellipsoidal, but if so, then the ellipsoidal coordinates (r, ϑ) from (3.14) are related to the (r, ϑ) above by a complicated transformation. Who will be able to find it or disprove its existence?

5. An Ellipsoidal Interior Kerr Solution — Some Remarks

As suggested above we should not expect that the ellipsoids r_{BL} = const in the Kerr metric describe possible surfaces of some sources. However, such a possibility was discussed by a few authors. This possibility may arise if either the source is not composed of a perfect fluid or the angular velocity of rotation (in the sense of Boyer [15]) is singular on the axis (in both cases the proof made by Roos [71] fails). Consequently it might be worthwhile to discuss this case at some length.

The fact that the ellipsoids which are supposed to be surfaces of some stars are confocal might be objected to by an astronomer. In such a confocal congruence, the smaller the ellipsoid is, the more it is flattened, while real stars tend rather to a reverse correlation [70]. This is a spurious contradiction. An astronomer, when speaking about small and big stars, has in mind objects of various masses and angular momenta. If the exterior gravitational fields of stars are described by the Kerr metric, then by a big (small) star an astronomer means a star with big (small) m necessitated by big (small) size. In our case we keep m and a fixed, and compare the shapes of stars with the same mass and angular momentum, differing *only* in size. It is clear that if a body is squeezed to a smaller size then to retain its initial angular momentum ma it must speed up its rotation. In such a situation, when $r \to 0$, the centrifugal force on the equator grows as r^{-3}, faster than the gravitational force, so the oblateness of the body must increase.

Since the ellipsoids are described by $r = r_0$ = const in a local subspace of both the Kerr metric and the interior metric, these metrics should be matched across the $r = r_0$ hypersurface, with r_0 allowed to vary in some finite range (limited by physical properties of matter, forbidding the star to be too small or too big). The ϑ-dependence of the two metrics on the bounding hypersurface should then be identical. Consequently, the most obvious guess for the interior metric is to assume that it all depends on ϑ in just the same way as the Kerr metric does. Such a guess was verified by the present author (unpublished). A trial metric was arranged which was of the form (3.14), and had unknown functions of r in all the places where the Kerr metric had constants or definite functions of r. For instance, g_{rr} from (4.2) was substituted by $[H(r) + \cos^2 \vartheta]/L(r)$, $g_{t\varphi}$ by $\{M(r) - P(r)/[N(r) + \cos^2 \vartheta]\}$, and so on. Then the metric was substituted in the field equations for a perfect fluid: $R^i{}_j = (\kappa/c^2)\, T^i{}_j - \frac{1}{2}\delta^i{}_j T) + \Lambda \delta^i{}_j$. Only those equations were used which had zero on the right-hand side (in this way no assumptions concerning ϵ and p were needed). They were easily arranged to be of the form $P(\cos^2 \vartheta) = 0$, where $P(\cdot)$ was a polynomial of 12th or 14th degree in $\cos^2 \vartheta$, and its coefficients were differential expressions in r. After the coefficients were equated to zero one by one, the whole problem reduced to a set of about 60 ordinary differential equations for 9 unknown functions of r. They were highly complicated, but nevertheless possible to solve with much effort. It appeared that the only metric of this kind is the Kerr metric itself. A perfect fluid source cannot be so similar to the Kerr solution; it must depend on ϑ in a more tricky way.

This conjecture is confirmed by the Demianski's solution (4.6) because the Λ-term imitates a primitive energy–momentum tensor somewhat similar to that of a perfect fluid. However, it should be verified if the simple guess would work for an energy–momentum tensor of mixed perfect fluid–electromagnetic field type, since the Kerr–Newman metric (4.2) is just of the guessed form.

A simplified version of the presented procedure (only m changed to an unknown function of r) was used by Gürses and Gursey [41], and by Collas and Lawrence [42] to construct a nonperfect fluid source of the Kerr metric (see Section 1).

The negative result described here explains at once the failure of Hernandez's efforts [17–18] (see also Section 1). He pursued essentially the same guess, but forced some of the functions of r to be different from those in the Kerr metric, and obtained a contradiction. This means of course that the guess was incorrect, not that the Kerr metric has no fluid source, as Hernandez tried to suggest. Nature is wiser than we are and if we find ourselves unable to solve some problem, this is a statement about our abilities, not about Nature. The author's opinion is that the efforts along the lines presented here should continue. Destructive statements denying the existence of a material source for the Kerr metric should be rejected until (if ever) they are reasonably justified.

Acknowledgments

I appreciate Dr. Wojciech Dziembowski's instruction about Newtonian figures of equilibrium. I am grateful to Professor A. Trautman and Dr. B. Mielnik for encouragement and useful comments which, among others, gave rise to the five Problems. Finally, I thank Dr. B. Kuchowicz for informing me about some unpublished references, and Drs. F. de Felice, P. S. Florides, V. P. Frolov, M. Gürses, P. A. Hogan, B. Léauté, and Miss B. Muchotrzeb for instructive correspondence and discussions concerning Section 1.

References

1. R. P. Kerr, *Phys. Rev. Lett.* **11** (1963), 237.
2. G. C. Debney, R. P. Kerr, and A. Schild, *J. Math. Phys.* **10** (1969), 1842.
3. R. H. Boyer and T. G. Price, *Proc. Cambridge Philos. Soc.* **61** (1965), 531.
4. J. Lense and H. Thirring, *Phys. Z.* **19** (1918), 156.
5. J. M. Cohen, *J. Math. Phys.* **9** (1968), 905.
6. B. Léauté, *Ann. Inst. H. Poincaré A* **8** (1968), 93.
7. M. K. Moss and W. R. Davis, *Nuovo Cimento B* **11** (1972), 84.
8. A. Komar, *Phys. Rev.* **113** (1959), 934.
9. A. I. Janis and E. T. Newman, *J. Math. Phys.* **6** (1965), 902.
10. E. T. Newman and A. I. Janis, *J. Math. Phys.* **6** (1965), 915.
11. R. P. Kerr, Gravitational collapse and rotation, *in* "Quasi-stellar Sources and Gravitational Collapse" (I. Robinson, A. Schild, and E. L. Schücking, Eds.), Chap. 9, p. 99, Univ. of Chicago Press, Chicago, 1965.
12. B. Léauté, "Trois études en relativité générale," These du doctorat, p. II-64, Université Pierre et Marie Curie, Paris, 1975.
13. L. Bel, *J. Math. Phys.* **10** (1969), 1501.
14. M. Misra, *Proc. Nat. Inst. Sci. India A* **26** (1960), 673.
15. R. H. Boyer, *Proc. Cambridge Philos. Soc.* **61** (1965), 527.
16. R. H. Boyer, *Proc. Cambridge Philos. Soc.* **62** (1966), 495.

17. W. C. Hernandez, *Phys. Rev.* **159** (1967), 1070.
18. W. C. Hernandez, *Phys. Rev.* **167** (1968), 1180.
19. E. Herlt, *Ann. Physik.* **24** (1970), 177.
20. W. Roos, *Gen. Rel. Grav.* **7** (1976), 431.
21. M. A. Abramowicz, J. P. Lasota, and B. Muchotrzeb, *Comm. Math. Phys.* **47** (1976), 109
22. K. S. Thorne, Relativistic stars, black holes and grativtational waves, *in* "General Relativity and Cosmology. Proceedings of the International School of Physics "Enrico Fermi", Course 47" (R. K. Sachs, Ed.), p. 260, Academic Press, New York, 1971.
23. F. de Felice, L. Nobili, and M. Calvani, *Astron. Astrophys.* **47** (1976), 309.
24. F. de Felice, The Kerr metric and its astrophysical consequences, *in* "Atti della XV Riunione della Società Astronomica Italiana, 8–9 Ottobre 1971" (supplemento al Vol. **43** della Mem. Soc. Astron. Ital.), p. 179, Baccini and Chiappi, Firenze, 1972.
25. M. A. Abramowicz, W. Arkuszewski, and B. Muchotrzeb, *Lett. Nuovo Cimento* **15** (1976), 477.
26. J. M. Cohen, *J. Math. Phys.* **8** (1967), 1477.
27. D. R. Brill and J. M. Cohen, *Phys. Rev.* **143** (1966), 1011.
28. V. de la Cruz and W. Israel, *Phys. Rev.* **170** (1968), 1187.
29. J. B. Hartle and K. S. Thorne, *Astrophys. J.* **153** (1968), 807.
30. J. D. McCrea, *Proc. Roy. Irish Acad. A* **73** (1973), 25.
31. J. L. Synge, *Proc. Roy. Irish Acad. A* **69** (1970), 11.
32. P. S. Florides and J. L. Synge, *Proc. Roy. Soc. A* **280** (1964), 459.
33. P. S. Florides, *Nuovo Cimento B* **13** (1973), 1.
34. W. Arkuszewski, W. Kopczyński, and V. N. Ponomariev, *Ann. Inst. H. Poincaré A* **21** (1974), 89.
35. C. B. G. McIntosh, *Comm. Math. Phys.* **37** (1974), 335.
36. P. S. Florides, *Nuovo Cimento B* **25** (1975), 251.
37. H. Keres, *Zh. Eksper. Teor. Fiz.* **52** (1967), 768.
38. W. Israel, *Phys. Rev. D* **2** (1970), 641.
39. C. Lanczos, *Ann. Physik.* **74** (1924), 518.
40. A. Ya. Burinskii, *Izv. Vyssh. Ucheb. Zaved. Fiz.* n° **8** (1974), 21.
41. M. Gürses and F. Gursey, *J. Math. Phys.* **16** (1975), 2385.
42. P. Collas and J. K. Lawrence, "Trapped Null Geodesics in a Rotating Interior Metric," preprint, California State University, Northridge, 1975.
43. R. H. Boyer and R. W. Lindquist, *J. Math. Phys.* **8** (1967), 265.
44. P. A. Hogan, *Lett. Nuovo Cimento* **16** (1976), 33.
45. V. Hamity, *Phys. Lett. A* **56** (1976), 77.
46. M. A. Abramowicz, private communication, 1975.
47. A. Papapetrou, *Ann. Inst. H. Poincaré A* **4** (1966), 83.
48. W. Kundt and M. Trümper, *Z. Physik.* **192** (1966), 419.
49. M. Trümper, *Z. Naturforsch. A* **22** (1967), 1347.
50. B. Schmidt, *Z. Naturforsch. A* **22** (1967), 1351.
51. B. Carter, *Comm. Math. Phys.* **17** (1970), 233.
52. H. Müller zum Hagen, *Proc. Cambridge Philos. Soc.* **71** (1972), 381.
53. C. W. Misner, K. S. Thorne, and J. A. Wheeler, "Gravitation," p. 240, W. H. Freeman, San Francisco, 1973.
54. L. P. Eisenhart, "An Introduction to Differential Geometry with Use of Tensor Calculus," p. 161, University Press, Princeton, N.J., 1947; "Riemannian Geometry," p. 92, University Press, Princeton, N.J., 1949.
55. R. A. Lyttleton, "The Stability of Rotating Liquid Masses," University Press, Cambridge, 1953.
56. S. Chandrasekhar, "Ellipsoidal Figures of Equilibrium," Yale University Press, New Haven/London, 1969.

57. D. M. ZIPOY, *J. Math. Phys.* **7** (1966), 1137.
58. T. MORGAN AND L. MORGAN, *Phys. Rev.* **183** (1969), 1097.
59. A. P. SLIVINSKII, *Izv. Vyssh. Ucheb. Zaved. Fiz.* n° **9** (1970), 33.
60. A. KOPPEL, *Izv. Vyssh. Ucheb. Zaved. Fiz.* n° **9** (1975), 29.
61. E. T. NEWMAN, E. COUCH, K. CHINNAPARED, A. EXTON, A. PRAKASH, AND R. TORRENCE, *J. Math. Phys.* **6** (1965), 918.
62. R. M. MISRA, *Proc. Roy. Irish Acad. A* **69** (1970), 39.
63. J. TIOMNO, *Phys. Rev. D* **7** (1973), 992.
64. E. T. NEWMAN AND J. WINICOUR, *J. Math. Phys.* **15** (1974), 426.
65. P. A. HOGAN, *Nuovo Cimento B* **29** (1975), 322.
66. P. A. HOGAN, *Proc. Roy. Irish Acad. A* **76** (1976), 37.
67. M. DEMIAŃSKI, *Acta Astron.* **23** (1973), 197.
68. V. P. FROLOV, *Teor. Mat. Fiz.* **21** (1974), 213.
69. B. CARTER, Black hole equilibrium states, *in* "Black holes—les astres occlus" (Proceedings of the 1972 course of the Les Houches School) (C. and B. S. de Witt, Eds.), p. 57, Gordon and Breach, New York/London/Paris, 1973.
70. YA. B. ZELDOVICH, private communication, 1975.
71. W. ROOS, "The Matter–Vacuum Matching Problem in General Relativity: General Methods and Special Cases," preprint, I. Institut für theoretische Physik, Universität Hamburg, October 1976.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.