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ROTATIONAL MOTION OF MATTER IN GENERAL RELATIVITY

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Abstract. It is argued that in the further development of cosmology the overidealized assumption leading to Friedman-Lemaître models should be gradually relaxed, and substituted by more realistic ones taking into account the departures of the real Universe from perfect homogeneity and isotropy. The paper presents the mathematical formalism for treating a class of irregular models: those with nonzero vorticity. A short account of history of the problem is given. Then the invariant types of motion of a continuous medium (expansion, rotation and shear) are presented, first in the Newtonian, and then in the relativistic hydrodynamics. The isentropic motion of a perfect fluid is dealt with in more detail. Plebański's formalism for treating the isentropic rotational motion of a perfect fluid is presented.

1. ROTATION AND COSMOLOGY

Among all cosmological models found up to now the homogeneous and isotropic models of Friedman-Lemaître appeared best. They describe very well the general, large-scale properties of the real Universe. However, they contain many unrealistic, highly idealized assumptions discussed in my former article. For instance, in an exactly homogeneous and isotropic Universe no galaxies may form, as the condensations of matter produced by random fluctuations of density grow too slowly to form sufficiently dense systems in the bounded time of 10^{10} years (Weinberg 1972, Jones 1976). Even if a galaxy is produced by an unexplained, artificially introduced perturbation then the other problem remains open: why do the observed galaxies rotate if the primordial matter of the Universe did not rotate? This problem led Gamow (1946) to the following suggestion published in a letter to "Nature": the rotation of stars is explained by the fact that they condensed out of a rotating galaxy.

also that the velocity of the fluid $v_j(x_i, t)$ is a well-defined differentiable function of the space-point and of time, thus the velocity is a vector field on the region of space considered. We have:

$$\frac{dx_j(t)}{dt} = v_j(x_i, t)\,, \tag{1}$$

where $x_j(t)$ on the left-hand side is a coordinate of a chosen particle of the fluid. (Note the difference: x_j is a function of time on the left because the motion is described by the sequence of values which the coordinates of the particle assume during motion. The x_j on the right does not depend on time: it is attached to a fixed point of space. The derivative on the left is calculated at that $x_j(t)$ which, in the moment t, coincides with the x_i on the right.)
Now let us choose a particle P of the fluid and assume that in the moment t it occupies the point x_i, $i = 1, 2, 3$. Let us choose another particle Q which, in the same moment t, occupies the nearby point $(x_i + \delta x_i)$ where δx_i is assumed small. The particle P has the velocity $v_j(x_i, t)$, the particle Q has in the same moment the velocity $v_j(x_i + \delta x_i, t)$. Consequently, exact to terms of first order in δx_i, the particle Q moves relative to the particle P with the velocity:

$$(v_{QP})_j = v_j(x_i + \delta x_i, t) - v_j(x_i, t) = v_{j,k}(x_i, t)\delta x_k + O(\delta^2)\,, \tag{2}$$

where the comma denotes the partial derivative, summation over repeated indices from 1 to 3 is assumed, and $O(\delta^2)$ denotes terms of the order of 2 and higher in δx_i.
It follows that at the moment $(t + dt)$ the particle Q will occupy, relative to the particle P, the position given by the vector:

$$\begin{aligned}\delta x_j' = \delta x_j + (v_{QP})_j dt + O(dt^2) &= \delta x_j + v_{j,k}\delta x_k dt + O(\delta^2, dt^2)\\ &= (\delta_{jk} + v_{j,k}dt)\delta x_k + O(\delta^2, dt^2)\,,\end{aligned} \tag{3}$$

where δ_{jk} is the Kronecker delta. We see from (3) that the matrix $v_{j,k}$ determines the velocity with which two nearby particles move relative to each other. If we decide to use solely cartesian coordinate systems, then the matrix $v_{j,k}$ may be split into three parts, each of which transforms independently of the two others when we change one cartesian frame to another:

$$v_{j,k} = \sigma_{jk} + \omega_{jk} + \tfrac{1}{3}\delta_{jk}\Theta\,, \tag{4}$$

where:

$$\Theta = v_{j,j} \tag{5}$$

is the trace of the matrix,

$$\sigma_{jk} = \tfrac{1}{2}(v_{j,k} + v_{k,j}) - \tfrac{1}{3}\delta_{jk}\Theta \tag{6}$$

is the trace-free symmetric part, and

$$\omega_{jk} = \tfrac{1}{2}(v_{j,k} - v_{k,j}) \tag{7}$$

is the antisymmetric part. This is a purely formal partition which may be done for any tensor of second rank. Here however each of the terms has a kinematic interpretation. This interpretation is easiest to see when each of the terms is singled out in its pure form. For this reason we shall consider three types of motion separately.

First lest us assume that $\omega_{jk} = \sigma_{jk} = 0$, $\Theta \neq 0$. Then the formula (3) yields:

$$\begin{aligned} \delta x_i' &= (\delta_{ik} + \tfrac{1}{3}\delta_{ik}\Theta dt)\delta x_k + O(\delta^2, dt^2) \\ &= (1+\tfrac{1}{3}\Theta dt)\delta x_i + O(\delta^2, dt^2)\,. \end{aligned} \tag{8}$$

The new vector $\delta x_i'$ has here the same direction as δx_i, but a different length, and $\frac{1}{3}\Theta$ is the logarithmic derivative of the length of δx_i with respect to time. Consequently, in this type of motion each two particles recede one from the other (or, if $\Theta < 0$, approach one another) along the straight line joining them. Such a type of motion is called isotropic expansion, and Θ is called the scalar of expansion.

Let us next consider such motion in which $\Theta = 0$, $\sigma_{ik} = 0 \neq \omega_{ik}$. Then:

$$\delta x_i' = (\delta_{ik} + \omega_{ik} dt)\delta x_k + O(\delta^2, dt^2)\,. \tag{9}$$

Let us find the length of the new vector of relative position:

$$\begin{aligned} \delta l' &= (\delta x_i' \delta x_i')^{1/2} \\ &= [(\delta_{ik} + \omega_{ik} dt)(\delta_{il} + \omega_{il} dt)\delta x_k \delta x_l + O(\delta^3, dt^2)]^{1/2} \\ &= [(\delta_{kl} + \omega_{kl} dt + \omega_{lk} dt)\cdot \delta x_k \delta x_l + O(\delta^3, dt^2)]^{1/2} \\ &= [\delta x_k \delta x_k + O(\delta^3, dt^2)]^{1/2} \\ &= (\delta x_k \delta x_k)^{1/2} + O(\delta^3, dt^2) = \delta l + O(\delta^3, dt^2)\,, \end{aligned} \tag{10}$$

where we have used the fact that $\omega_{kl} + \omega_{lk} = 0$. Consequently, the length of the vector δx_i, exact to terms of order $O(\delta^2, dt)$, does not change in this type of motion at all. To see what happens with the direction of δx_i let us take the increment $(\delta x_i' - \delta x_i)$ and find its scalar product with δx_i:

$$(\delta x_i' - \delta x_i)\delta x_i = \omega_{ik} dt\, \delta x_k \delta x_i + O(\delta^3, dt^2) = 0 + O(\delta^3, dt^2)\,, \tag{11}$$

where again antisymmetry of ω_{ik} was used. So with the same precision, the increment of δx_i is perpendicular to δx_i. The properties (10) and (11) are characteristic for pure rotational motion, so in this case two nearby particles rotate around each other. The term ω_{ik} is called the rotation tensor.

Let us see what the angular velocity of their rotation is. Let the searched angular velocity vector be $\boldsymbol{\omega}$. We have:

$$\mathbf{v}_{QP} = \boldsymbol{\omega} \times \boldsymbol{\delta}\mathbf{x} \tag{12}$$

in other words:

$$(v_{QP})_i = \varepsilon_{ikl}\omega_k \delta x_l\,. \tag{13}$$

But:

$$(v_{QP})_i = \omega_{il}\delta x_l \,. \tag{14}$$

Hence:

$$\omega_{il}\delta x_l = \varepsilon_{ikl}\omega_k\delta x_l \,. \tag{15}$$

This is an identity in δx_l, consequently:

$$\omega_{il} = -\varepsilon_{ilk}\omega_k \,. \tag{16}$$

Reversing this and using the formula (7) we obtain:

$$\omega_i = \tfrac{1}{2}\varepsilon_{ikl}\omega_{lk} = \tfrac{1}{2}\varepsilon_{ikl}v_{l,k} \tag{17}$$

what means:

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\mathrm{rot}\,\mathbf{v} \,. \tag{18}$$

The equations (16) and (17) establish the unique correspondence between the angular velocity vector $\boldsymbol{\omega}$ and the rotation tensor ω_{ik}. This justifies the name of ω_{ik}.

Finally, let us take the case $\Theta = 0$, $\omega_{ik} = 0 \neq \sigma_{ik}$. Let us consider four nearby particles, and three vectors $\boldsymbol{\delta}\mathbf{x}$, $\boldsymbol{\delta}\mathbf{y}$, $\boldsymbol{\delta}\mathbf{z}$ joining one of the particles to the three others. Then, by an easy but tedious calculation it may be shown that the quantity $\varepsilon_{ijk}\delta x_i \delta y_j \delta z_k = \boldsymbol{\delta}\mathbf{x}\cdot(\boldsymbol{\delta}\mathbf{y}\times\boldsymbol{\delta}\mathbf{z})$ is preserved during the motion. This means that the volume of the paralellogram spanned by $\boldsymbol{\delta}\mathbf{x}$, $\boldsymbol{\delta}\mathbf{y}$ and $\boldsymbol{\delta}\mathbf{z}$ is constant during the motion, though its shape is in general not preserved

4. THE MOTION OF A CONTINUOUS MEDIUM IN RELATIVISTIC HYDRODYNAMICS

The motion of a fluid in general relativity is described similarly as in Newtonian theory. We assume that through every point x^α of a region of spacetime there passes the world line of one particle of the fluid with a well-defined four-velocity $u^\alpha(x^\beta)$, where:

$$\frac{dx^\alpha}{ds} = u^\alpha(x^\beta) \tag{19}$$

s being the proper time on the world-line considered, and $x^\alpha(s)$ are the coordinates of the flowing particle. (Note the remark after formula (1), it applies here with t changed to s, and "time" changed to "proper time".) Since u^α is tangent to the world-line, all the vectors perpendicular to world-lines obey the equation:

$$g_{\alpha\beta}u^\alpha A^\beta = 0 \,. \tag{20}$$

Notice now that the matrix:

$$h^\alpha_\beta \stackrel{\text{def}}{=} \delta^\alpha_\beta - u^\alpha u_\beta \tag{21}$$

projects all the vectors onto hypersurfaces locally orthogonal to u^α, i.e. for every four-vector B^α the vector $B^\alpha_\perp \stackrel{\text{def}}{=} h^\alpha_\beta B^\beta$ obeys the equation $g_{\alpha\beta} u^\alpha B^\beta_\perp = 0$. Moreover, $h^\alpha_\beta u^\beta = 0$, and $g_{\alpha\beta} B^\alpha_\perp B^\beta_\perp = h_{\alpha\beta} B^\alpha_\perp B^\beta_\perp$ which means that $h_{\alpha\beta}$ acts as the metric tensor in the hypersurface locally orthogonal to a world-line.

If we choose the lines of the time-coordinate x^0 so that they coincide with the stream-lines of the fluid (what is always possible) then the condition $h^\alpha_\beta u_\beta = 0$ means that x^0 remains constant when we stay within the hypersurface locally orthogonal to u^α. Therefore, for this specific choice of time, that hypersurface contains the events simultaneous with the event on which it intersects with its orthogonal u^α-line. Let us remember however that in general (just when u^α has nonzero rotation as defined below) the hypersurface is orthogonal to only one world-line, and consequently it defines the collection of events simultaneous with the given one only in a small neighbourhood of that event.

Now let us choose one world-line P, and let P_0 be the point of P to which our particle following P arrives in the moment s of its proper time. Let Q_0 be a point not too distant from P_0, lying on a neighbouring world-line Q, and let δx^α be the four-vector joining P_0 to Q_0. The event of line Q simultaneous with P_0 will be joined from P_0 by the vector:

$$\delta_\perp x^\alpha = h^\alpha_\beta(P_0)\,\delta x^\beta\,. \tag{22}$$

The vector $\delta_\perp x^\alpha$ gives consequently the spacelike orientation of two nearby particles of the fluid. The velocity of the point P_0 is $u^\alpha(x^\beta)$, the velocity of the point $(x^\beta + \delta_\perp x^\beta)$ on the line Q is $u^\alpha(x^\beta + \delta_\perp x^\beta)$. Thus after the time ds the particle which was at the point $P_0 = \{x^\alpha\}$ in the moment s will move to the point $x'^\alpha = x^\alpha + u^\alpha(x^\beta)\,ds$, while the corresponding particle on line Q which was at the point $(x^\alpha + \delta_\perp x^\alpha)$ will move to the point:

$$x''^\alpha = x^\alpha + \delta_\perp x^\alpha + [u^\alpha(x^\beta + \delta_\perp x^\beta)]^{\|}_{x^\beta + \delta_\perp x^\beta \to x^\beta}\, ds\,. \tag{23}$$

The symbol $[\ \ ]^{\|}_{x^\beta + \delta_\perp x^\beta \to x^\beta}$ denotes the parallel displacement of a vector attached at the point $(x^\beta + \delta_\perp x^\beta)$ to the point (x^β) (only then it can be added to the vector $\delta_\perp x^\alpha$ attached at the point (x^β), otherwise, as is well known from differential geometry, adding two vectors attached each to a different point would make no sense). It is known that:

$$[u^\alpha(x^\beta + \delta_\perp x^\beta)]^{\|}_{x^\beta + \delta_\perp x^\beta \to x^\beta} = u^\alpha(x^\beta) + u^\alpha{}_{;\varrho}\,\delta_\perp x^\varrho + O(\delta^2)\,, \tag{24}$$

where the semicolon denotes a covariant derivative. Thus in the moment $(s + ds)$ the event on line Q simultaneous to the event x'^α from the line P will occupy relative to x'^α the following position:

$$\begin{aligned}\delta_\perp x'^\alpha = x''^\alpha - x'^\alpha &= \delta_\perp x^\alpha + u^\alpha{}_{;\varrho}\,\delta_\perp x^\varrho ds + O(\delta^2)\\ &= (\delta^\alpha_\varrho + u^\alpha{}_{;\varrho})\, ds\, \delta_\perp x^\varrho + O(\delta^2)\,.\end{aligned} \tag{25}$$

However, not the whole matrix $u^{\alpha}{}_{;\varrho}$ effectively enters the formula (25). Let us make the following substitution (which is just an identity):

$$u^{\alpha}{}_{;\varrho} = u^{\alpha}{}_{;\sigma}\delta^{\sigma}_{\varrho} = u^{\alpha}{}_{;\sigma}(h^{\sigma}_{\varrho} + u^{\sigma}u_{\varrho}) \,. \tag{26}$$

We see that only the first term in parentheses gives nonzero contribution to (25) since $u_{\varrho}\delta_{\perp}x^{\varrho} \equiv 0$. Therefore:

$$\delta_{\perp}x'^{\alpha} = (\delta^{\alpha}_{\varrho} + u^{\alpha}{}_{;\sigma}h^{\sigma}_{\varrho}ds)\,\delta_{\perp}x^{\varrho} + O(\delta^2) \,. \tag{27}$$

Comparing this last formula with (3) we see that the matrix $u^{\alpha}{}_{;\sigma}h^{\sigma}_{\varrho}$ plays the same role in relativistic hydrodynamics as the matrix $v_{j,k}$ did in Newtonian hydrodynamics. Now the same partition into three independent parts may be done:

$$u_{\alpha;\varrho}h^{\varrho}_{\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \tfrac{1}{3}\Theta h_{\alpha\beta} \,, \tag{28}$$

where:

$$\Theta = u^{\sigma}{}_{;\varrho}h^{\varrho}_{\sigma} \equiv u^{\varrho}{}_{;\varrho} \tag{29}$$

$$\omega_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha;\varrho}h^{\varrho}_{\beta} - u_{\beta;\varrho}h^{\varrho}_{\alpha}) \tag{30}$$

$$\sigma_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha;\varrho}h^{\varrho}_{\beta} + u_{\beta;\varrho}h^{\varrho}_{\alpha}) - \tfrac{1}{3}\Theta h_{\alpha\beta} \,. \tag{31}$$

The three terms have here exactly the same interpretation as the corresponding Newtonian terms denoted by the same letters in formulae (5)—(7). The rotation tensor has here the following explicit form:

$$\omega_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha;\beta} - u_{\beta;\alpha} - u_{\alpha;\varrho}u^{\varrho}u_{\beta} + u_{\beta;\varrho}u^{\varrho}u_{\alpha}) \equiv \tfrac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha} - \dot{u}_{\alpha}u_{\beta} + \dot{u}_{\beta}u_{\alpha}) \,, \tag{32}$$

where the vector $\dot{u}^{\alpha} \overset{\text{def}}{=} u^{\alpha}{}_{;\varrho}\, u^{\varrho}$ is called the acceleration vector. We have substituted partial derivatives for covariant ones because in such an antisymmetric combination the connection terms cancel out.

5. ISENTROPIC MOTION OF A PERFECT FLUID

The energy-momentum tensor of a perfect fluid has the form:

$$T^{\beta}_{\alpha} = (\varepsilon + p)u_{\alpha}u^{\beta} - p\delta^{\beta}_{\alpha} \,, \tag{33}$$

where ε is the energy density and p is the pressure of the fluid. The quantity $(\varepsilon + p)$ is the enthalpy density. This tensor should fulfil the following equations of motion:

$$T^{\beta}_{\alpha\,;\beta} = 0 \,. \tag{34}$$

Independently of (34) we assume that in addition the total rest-mass of particles of the fluid is constant, i.e. no particles are created or destroyed. This assumption is specified by the additional equation:

$$(\varrho u^\beta)_{;\beta} = 0\,, \tag{35}$$

where ϱ is the rest-mass density.
Let us introduce the following definition:

$$H = (\varepsilon + p)/\varrho c^2\,. \tag{36}$$

It is a dimensionless quantity which equals the enthalpy per unit rest energy of the fluid. Using (33), (36) and (35) we may write equation (34) in the form:

$$\varrho c^2 u^\beta (Hu_\alpha)_{;\beta} - p_{,\alpha} = 0\,. \tag{37}$$

We shall now assume that the fluid is chemically homogeneous, i.e. that it is composed of identical particles. Then $\varrho c^2/n = m_0 c^2 = \text{const}$ where n is the density of number of particles, and m_0 is the mass of one particle. In this case the enthalpy per unit rest energy, H, is proportional (with the constant coefficient $n/\varrho c^2$) to the enthalpy per unit particle which will be denoted by $\mathcal{H}$. This other quantity obeys, in the conventional nonrelativistic thermodynamics, the following identity:

$$d\mathcal{H} = Vdp + Td\sigma = n^{-1}dp + Td\sigma\,, \tag{38}$$

where $V = n^{-1}$ is the volume occupied by one particle, and σ is the entropy per unit particle. Consequently:

$$dH = (\varrho c^2)^{-1}dp + TdS\,, \tag{39}$$

where this time the entropy S is calculated per unit rest energy. The equation (39) may be taken as the definition of temperature and entropy in general relativity (see details in Krasiński 1974). Hence we have:

$$p_{,\alpha} = \varrho c^2 (H_{,\alpha} - TS_{,\alpha})\,. \tag{40}$$

Substituting (40) in (37) we obtain:

$$\varrho c^2 [u^\beta (Hu_\alpha)_{;\beta} - H_{,\alpha} + TS_{,\alpha}] = 0 \tag{41}$$

Now we can insert the factor $u^\beta u_\beta = 1$ by $H_{,\alpha}$, and the additional term $-Hu^\beta u_{\beta;\alpha} \equiv 0$ in parentheses. Then (41) reads, after cancelling ϱc^2:

$$\begin{aligned} O &= u^\beta[(Hu_\alpha)_{;\beta} - H_{,\alpha} u_\beta - Hu_{\beta;\alpha}] + TS_{,\alpha} \\ &= u^\beta[(Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}] + TS_{,\alpha}\,. \end{aligned} \tag{42}$$

This form of the equations of motion is equivalent to (34). We call the motion isentropic when $S = \text{const}$. Then:

$$[(Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}]u^\beta = 0\,. \tag{43}$$

Then from (39) we also conclude that H and ϱ are functions of p, and:

$$d\left(\frac{\varepsilon+p}{\varrho c^2}\right) \equiv dH = \frac{dp}{\varrho c^2}. \tag{44}$$

Integrating this equation from 0 to p, and assuming the initial condition $\varepsilon(0) = \varrho(0)c^2$ we obtain:

$$H = 1+\frac{1}{c^2}\int\limits_0^p \frac{d\tilde{p}}{\varrho(\tilde{p})}. \tag{45}$$

The equations (43) and (45) describe the isentropic motion of a perfect fluid. In the case of dust (matter interacting only gravitationally) we have $p = 0$ and $H = 1$. Then the equations of motion (43) follow directly from (34) and assume the simple form:

$$(u_{\alpha,\beta}-u_{\beta,\alpha})u^\beta = 0. \tag{46}$$

6. ISENTROPIC ROTATIONAL MOTION

The equations (43) admit as a special solution:

$$(Hu_\alpha),_\beta-(Hu_\beta),_\alpha = 0. \tag{47}$$

Such motion is called irrotational. If however (47) is not fulfilled then the solution of (43) is less trivial, and the motion is called rotational. It may be shown (Krasiński 1974) that then:

$$(Hu_\alpha),_\beta-(Hu_\beta),_\alpha = 2H\omega_{\alpha\beta}, \tag{48}$$

where $\omega_{\alpha\beta}$ is defined by (32). Consequently, the distinction between rotational and irrotational motions introduced above is consistent with that based on kinematics.

When (47) is not fulfilled then it may be shown that the equations (43) imply the existence of three scalar functions (τ, ξ, η) such that:

$$Hu_\alpha = \tau,_\alpha+\eta\xi,_\alpha \tag{49}$$

(Krasiński 1974). Then:

$$2H\omega_{\alpha\beta} = \xi,_\alpha\eta,_\beta-\xi,_\beta\eta,_\alpha. \tag{50}$$

Since the gradients $\xi,_\alpha$ and $\eta,_\alpha$ are linearly independent (otherwise (47) would be fulfilled, contrary to our assumption), the equations (43) and (48) imply:

$$u^\alpha\xi,_\alpha = u^\alpha\eta,_\alpha = 0. \tag{51}$$

We see that $(u^\alpha, \xi_{,\alpha}, \eta_{,\alpha})$ form a triple of linearly independent vectors. Consequently, a fourth vector ζ_δ may be adjusted to them so that:

$$(-g)^{1/2}\varrho u^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\xi_{,\beta}\eta_{,\gamma}\zeta_\delta\,, \tag{52}$$

where g is the determinant of the metric tensor, and $\varepsilon^{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita symbol. So far ζ_δ is just a vector obeying (52). With use of the equation (35) rewritten in the form $(-g)^{-1/2}[(-g)^{1/2}\varrho u^\alpha]_{,\alpha} = 0$ it may be easily shown (Krasiński 1974) that the vector ζ_δ may be chosen so that it is equal to the gradient of a scalar function which we shall also denote by ζ, i.e. $\zeta_\delta = \zeta_{,\delta}$. Consequently:

$$(-g)^{1/2}\varrho u^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\xi_{,\beta}\eta_{,\gamma}\zeta_{,\delta}\,. \tag{53}$$

We see that the gradient of ζ is linearly independent of u^α, $\xi_{,\alpha}$ and $\eta_{,\alpha}$, so the function ζ is independent of τ, ξ, η. We have therefore four independent functions (τ, ξ, η, ζ) which can be used as new coordinates:

$$x^0 = \tau\,, \quad x^1 = \xi\,, \quad x^2 = \eta\,, \quad x^3 = \zeta\,. \tag{54}$$

They are determined up to the following transformations (Krasiński 1974):

$$\begin{aligned} x^0 &= x^{0'} - S(x^{1'}, x^{2'}) \\ x^1 &= F(x^{1'}, x^{2'}) \\ x^2 &= G(x^{1'}, x^{2'}) \\ x^3 &= x^{3'} + T(x^{1'}, x^{2'})\,, \end{aligned} \tag{55}$$

where the function T is arbitrary, while F and G are connected by the equation:

$$F_{,1'}G_{,2'} - F_{,2'}G_{,1'} = 1\,. \tag{56}$$

The function S is determined by F and G through the equations:

$$S_{,1'} = GF_{,1'} - x^{2'}\,, \qquad S_{,2'} = GF_{,2'}\,. \tag{57}$$

In these coordinates the equation (53) assumes the form:

$$(-g)^{1/2}\varrho u^\alpha = \delta^\alpha_0 \tag{58}$$

the equation (49) goes over into:

$$u_\alpha = (\delta^0_\alpha + x^2\delta^1_\alpha)H^{-1} \tag{59}$$

and taking the scalar product of the equations (58) and (59) we obtain, by virtue of $u^\alpha u_\alpha = 1$:

$$(-g)^{1/2}\varrho H = 1\,. \tag{60}$$

Consequently:

$$g = -\varrho^{-2}H^{-2} \tag{61}$$

and (58) yields:

$$u^\alpha = H\delta^\alpha_0\,. \tag{62}$$

But $u_a = g_{a\varrho} u^\varrho$, so (59) and (62) imply:

$$g_{00} = H^{-2}, \quad g_{01} = x^2 H^{-2}, \quad g_{02} = g_{03} = 0\,. \tag{63}$$

In these coordinates the relativistic vorticity vector

$$w^a \stackrel{\text{def}}{=} -(-g)^{-1/2} \varepsilon^{a\beta\gamma\delta} u_\beta u_{\gamma,\delta}$$

assumes the form:

$$w^a = \varrho H^{-1} \delta^a_3\,. \tag{64}$$

The method of description of isentropic rotational motion presented here (introduced by J. Plebański, 1970) was used to obtain a series of new solutions of the Einstein field equations (Krasiński 1974 and 1975). Unfortunately, they were again nonrealistic because of their stationarity. The coordinates defined above are well suited for comparing existing solutions because there exists an explicit prescription for constructing these coordinates. However, they have two defects:

1. The otherwise simple solutions, when represented in the coordinates (61)—(63) become in general rather complex.
2. These coordinates do not exist for a fluid moving irrotationally, and so one cannot pass to the limit of zero rotation without changing to another frame.

For these reasons it is not clear if the method presented here will prove useful for cosmology. An attempt to use it for constructing a realistic model of a rotating Universe will be presented in my next article. A success by that attempt would be the first physically interesting result obtained by this method.

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