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CYLINDRICAL ROTATING UNIVERSE

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Abstract. The method presented in the author's other article in this issue is applied to the special case of cylindrically symmetric rotating and expanding cosmological models. It is argued that they are the simplest rotating generalizations of the Friedman models which contain no additional, independent of matter geometrical structures, as opposed to the Bianchi-type models. The assumed symmetries of spacetime, together with the simplest field equations are used for simplifying the metric tensor as much as possible. The paper ends in the place where new simplifying assumptions appear necessary for further integration of the field equations.

1. WHY SHOULD ONE CONSIDER CYLINDRICAL MODELS WITH ROTATION?

In the beginning let us recall two facts discussed in my other article on cosmological models in this issue:

1. The observational support for the assumptions of Friedman-Lemaître models is in general very inexact and uncertain.
2. Yet the Friedman models are the best of all existing models: they lead to results satisfactorily consistent with observations, and they even allowed the prediction of at least one effect (the relic radiation) before its discovery became commonly known.

The first fact implies that we should not content ourselves with Friedman models, but look for new models founded on less idealized assumptions which might eventually explain why the Universe is so highly regular (if it really is). The second fact shows however that the departure away from

Friedman models should not be too radical. The natural first step will then be to look for such models which violate the assumptions of spacelike homogeneity and isotropy as little as possible. For this reason much care has recently been devoted to the homogeneous Bianchi-type models. They violate only the assumption of isotropy, so they could seem to be the most natural generalization of the Friedman models. In some of the Bianchi models the matter may rotate. However, when rotation is present, the term "homogeneity" should be treated with some caution, since then it has a different physical meaning than in the Friedman models. In all the Bianchi models there exists a distinguished vector field e^0 which defines its orthogonal hypersurfaces. These hypersurfaces are just the homogeneous spaces. In the Friedman-like models the velocity field of matter is colinear with e^0 which means that the homogeneous space is at the same time the rest space of matter. No geometrical structures independent of matter are needed to reveal the spatial homogeneity of the spacetime. In models with rotation the velocity of matter cannot be colinear with e^0, since e^0 is irrotational being orthogonal to a congruence of spaces. It means, there exist in such models God-given preferred to certain observers for whom the Universe looks especially simple. It is not a decisive argument against those models, but this observation shows that two different properties were categorized by the same label "homogeneity", and that this word should not be used mechanically. If we take a portion of matter described by the Friedman model, and set it into rotation, then it is by no means obvious that a Bianchi-type model should result.

In the present article a new class of cosmological models will be presented: the cylindrically symmetric rotating Universes. They are free of the aforementioned drawback, while on the other hand they also violate the simple assumptions of Friedman models only to a moderate degree. Their bad feature is that they do not encompass the closed model, as will be shown in section 2.

In the rotating matter no isotropy may survive, since at every point of spacetime there is the distinguished direction of the vortex. The only remnant of isotropy may be the axial symmetry around the vorticity vector. The homogeneity along the direction of the vortex may be assumed a remnant of the 3-dimensional homogeneity of the Friedman models. Consequently, we have here only two Killing vector fields.

The model of the Universe obtained under such assumptions will not be fully satisfactory, as the galaxies produced in it would have their axes of rotation aligned, while real galaxies apparently do not (Jones 1976). Neither is it obvious that such a model would at all permit a description of galaxy formations in a way better than Friedman models do (where in principle no galaxies may form, Jones 1976). However, it is better to have a model in which the galaxies rotate parallelly than a model forbidding rotation at all.

2. CYLINDRICALLY SYMMETRIC SPACETIMES

The preferred coordinates introduced in my lecture about rotational motion will be used here.
We shall assume that cylindrical symmetry is characterized by the following four properties:

A. There exist two Killing vector fields $\underset{(\varphi)}{k}$ and $\underset{(z)}{k}$ of which $\underset{(\varphi)}{k}$ corresponds to axial symmetry, and $\underset{(z)}{k}$ generates translations parallel to the symmetry axis;

B. $\underset{(z)}{k}^{\mu} = f(x)\delta_3^{\mu}$, where $f(x)$ is a function, possibly of all four coordinates;

C. $[\underset{(z)}{k}, \underset{(\varphi)}{k}] = 0$;

D. $g_{\mu\nu}\underset{(z)}{k}^{\mu}\underset{(\varphi)}{k}^{\nu} = 0$.

The properties A, C, and D need no explanation as they simply reflect the symmetries of a cylindrical surface from Euclidean space. The property B, in virtue of eq. (64) from my former lecture, means that the symmetry axis coincides with the rotation axis.
We can now easily see that of the Bianchi models only those may be cylindrically symmetric which contain a two-parametric abelian subgroup whose generators obey condition D. Only the models of types VIII and IX do not contain abelian subgroups (Collins and Hawking 1973). There is no reason to worry about type VIII, but unfortunately type IX contains the closed Friedman model. Thus we see that the closed Friedman model generalized for rotation cannot be handled by the method presented here, it was excluded by assumption C.
It may be shown rather easily that if a metric tensor is preserved by some symmetry transformation, then so is the Riemann and Ricci tensor, and the curvature scalar, consequently also the energy-momentum tensor. Now, pressure and density of matter are eigenvalues of the energy-momentum tensor of a perfect fluid, so they are invariant under the same symmetry transformation, too. It follows that the four-velocity of the fluid is invariant, being an eigenvector of the energy-momentum tensor, and the vorticity vector is invariant, being invariantly constructed from the velocity vector. Thus, if we deal with the field equations for a perfect fluid, we have in addition to the properties A—D:

$$\underset{(\varphi)}{k}^{\mu}\varrho_{,\mu} = \underset{(\varphi)}{k}^{\mu}H_{,\mu} = \underset{(z)}{k}^{\mu}\varrho_{,\mu} = \underset{(z)}{k}^{\mu}H_{,\mu} = 0\,, \tag{1}$$

where H is defined by eq. (45) in my former lecture, and:

$$[\underset{(\varphi)}{k}, u] = [\underset{(\varphi)}{k}, w] = [\underset{(z)}{k}, u] = [\underset{(z)}{k}, w] = 0\,, \tag{2}$$

where u and w are the velocity and vortex vectors respectively.

Using these equations, together with some Killing equations and the properties B and C one can show that our special coordinates from the former lecture may be further specialized so that:

$$\underset{(\varphi)}{k}^{\mu} = \delta^{\mu}_{1}\,, \tag{3}$$

$$f = f(x^2)\,. \tag{4}$$

The detailed proof is easy, but long and involved, so we shall omit it here (it will be published elsewhere, see Krasiński 1976). In such a coordinate system the property D simply means:

$$g_{13} = 0\,. \tag{5}$$

Equation (3) implies that in such a coordinate system the metric tensor simply does not depend on x^1. Equation (4) shows that if one performs the transformation:

$$\begin{aligned} x^i &= x^{i'},\ i = 1, 2, 3, \\ x^3 &= f(x^2)\, x^{3'} \end{aligned} \tag{6}$$

then the transformed metric will not depend on x^3, as $\underset{(z)}{k}^{\mu} = f(x^2)\delta^{\mu}_{3}$ turns after such a transformation into $\underset{(z)}{k}^{\mu'} = \delta^{\mu'}_{3'}$. The transformation (6) does not belong to the class given by equations (55)—(57) in my former article, so it will result in a change of some properties of the metric tensor specified by the equations (61)—(63) there. Namely, after this transformation the determinant of the metric tensor will be equal to:

$$g = -f^2 \varrho^{-2} H^{-2} \tag{7}$$

while the vorticity vector will change to:

$$w^{\alpha} = \varrho f^{-1} H^{-1} \delta^{\alpha}_{3}\,. \tag{8}$$

So formally the result of transformation (6) is equivalent to the substitution:

$$\varrho \to \bar{\varrho} \overset{\text{def}}{=} \varrho / f \tag{9}$$

made on the left-hand side of the field equations.
The coordinates in which the equations (3—5) and (7—8) hold are specified up to the following transformations:

$$\begin{aligned} x^0 &= x^{0'} - C \int x^{2'} F_{,2'}\, dx^{2'} \\ x^1 &= C^{-1} x^{1'} + F(x^{2'}) \\ x^2 &= C x^{2'} \\ x^3 &= x^{3'} + T(x^{2'})\,, \end{aligned} \tag{10}$$

where F and T are arbitrary functions of one variable, and C is an arbitrary constant.

Up to this place the Einstein field equations were not explicitly used (except for the statement of invariance of ϱ, H, u and w). Three of them can now be relatively easy integrated (see details in Krasiński 1976) giving the result:

$$g_{23} = \alpha(x^2) g_{33}\,, \tag{11}$$

where $\alpha(x^2)$ is an arbitrary function of one variable. Now it is possible to perform transformation (10) with:

$$T(x^2) = -\int \alpha(x^2) dx^2 \tag{12}$$

to obtain:

$$g_{23} = 0 \tag{13}$$

in the new coordinates. After such a transformation the remaining freedom of choice of the cooidinates is given by (10) with $T = \text{const.}$

Thus we have shown that cylindrically symmetric rotating models considered here have the following metric form:

$$ds^2 = H^{-2}(dx^0)^2 + 2x^2 H^{-2} dx^0 dx^1 + g_{11}(dx^1)^2 + \\ + 2g_{12} dx^1 dx^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2\,, \tag{14}$$

where all the components of the metric depend only on x^0 and x^3, and the equation (7) holds.

The remaining, unsolved field equations are so complex that it is rather hopleess to integrate them without any additional simplifying assumptions. The simplest and self-suggesting assumption is a shearfree motion of matter, $\sigma_{\alpha\beta} = 0$. Indeed, the Einstein field equations appear possible to integrate in this case, but the result is of no interest for cosmology, as the solution is stationary, having no expansion. The corresponding stationary solutions were found and investigated (Krasiński 1974 and 1975). Therefore one could try to find the nonstationary solutions by inserting unknown functions of time in place of arbitrary constants from the stationary ones. This is however a purely calculational guess, with no physical justification.

It is at present an open problem to construct an explicit, completely solved model of an expanding and rotating Universe. Its finding could supply useful information about the behaviour of the Universe near the initial singularity. It is hoped that the paper presented here is a first step towards this aim.

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