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SOLUTIONS OF THE EINSTEIN FIELD EQUATIONS FOR A ROTATING PERFECT FLUID

Part 3:

A survey of models of rotating perfect fluid or dust

ABSTRACT

The article reviews models of rotating matter considered by other authors. The papers using approximate methods are only listed, while all the exact solutions of the field equations with rotating perfect fluid or dust as a source, published up to the end of 1973, are critically reviewed. It is shown, which of them are special cases of the solutions found in part 1 of the article, and which are not. In the second case the difference is explained. Each time when the data of the paper made it possible, the solutions were transformed to the "canonical coordinate system" used in part 1. A table exhibiting all the discoveries and rediscoveries of solutions under consideration is annexed. 60 refs.

INTRODUCTION

The present article constitutes the third part of a series of papers [1] , [2] on solutions of the Einstein field equations for a rotating perfect fluid. It will be referred to as chapter 14 of the whole work. There are no new results in it. I give here a review of solutions obtained by other authors. The references dealing with the approximate solutions are merely listed. A critical review of exact models of rotating perfect fluid or dust is given. This "neglectfulness" of approximate solutions needs justification.

Approximate methods may be quite useful when some effects in the gravitational field of a rotating body are concerned, as e.g. the shape of its surface, the non-flat deflection of a light-ray in the exterior field, the dragging of inertial frames, and so on. However, if one intended to use approximate solutions to investigate such properties as geodesic completeness, singularities or equation of state, then he might be lead to completely wrong conclusions. As an example, let us take the two-dimensional Riemannian metric:

$$ds^2 = \left(1 + e^{-\omega x^2}\right)^{-1} \left(dt^2 - dx^2\right)$$

One can hardly imagine a more regular space. It is infinite, geodesically complete, nonsingular, and has the signature $(+ -)$ for all values of x. Now take the first approximation in ω. Then it turns out that at $x = \pm (2/\omega)^{1/2}$ a singularity appears, which is at the same time a boundary of the space and no line can cross it. Take the second approximation in ω. The singularity disappears, but the line of the x - coordinate, which is still a geodesic, has again a finite length. If we take higher approximations, then the singularity appears in every odd step, but the x - line is always finite. Now, if one had only the approximate results at his disposal, would he recognize how regular the exact metric is?

There are still more objections against the method of slow rotation which is the most frequently used approximate method. In newtonian physics the definition of angular velocity is based on the notion of a radius-vector. This notion makes no sense in a general curved space, so the angular velocity is defined only for infinitesimally close points. The relative angular velocity of distant points is clearly not an invariant notion, and moreover it is by no means obvious that the linear connection between the linear velocity V and angular velocity ω should hold in general relativity. Most of authors do not care for these difficulties.

Since we are interested in geometrical properties of space-time, we shall not include the approximate solutions into our investigation. The list of references concerning approximate solutions [3] - [30] is given for completeness. The methods of numerical integration of the Einstein field equations for rotating matter, developed recently by M.P. Ryan jr. [31] and J.Pachner [32], [33] deserve a special care, but they also are outside the scope of our formalism.

For clarity of the paper it is necessary to summarize briefly the starting point and some of the results of part 1 [1]. We dealt with isentropic perfect fluid obeying the equation of continuity, or dust. Under the assumption of nonzero rotation the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ implied the existence of such coordinates, in which:

$$g_{00} = H^{-2} \qquad /14.1/$$

$$g_{01} = x^2 H^{-2} \qquad /14.2/$$

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$$g_{00} = H^{-2} \qquad /14.1/$$

$$g_{01} = x^2 H^{-2} \qquad /14.2/$$

$$g_{02} = g_{03} = 0 \qquad /14.3/$$

$$g = \det \|g_{\alpha\beta}\| = -\varrho^{-2} H^{-2} \qquad /14.4/$$

where $\varrho = \varrho(p)$ is the density of the rest-mass, and:

$$H \overset{\text{def}}{=} H_0 + \frac{1}{c^2}\int_0^p \frac{dp}{\varrho(p)} = (\epsilon + p)/\varrho c^2 \qquad /14.5/$$

The function H is the enthalpy per unit rest-energy, ϵ is the energy - density and H_0 = const. In the case of dust p = 0 and H = 1.

Moreover, we obtained:

$$u^\alpha = H \delta^\alpha_0 \qquad /14.6/$$

for the velocity field, and:

$$w^\alpha = \varrho H^{-1} \delta^\alpha_3 \qquad /14.7/$$

for the vorticity field. Next, we have assumed that

$$\frac{\partial}{\partial x^0} g_{\alpha\beta} = \frac{\partial}{\partial x^3} g_{\alpha\beta} = 0 \qquad /14.8/$$

what implies $\partial_u g_{\alpha\beta} = \partial_w g_{\alpha\beta} = 0$. The solutions of the field equations were then determined without any additional assumptions, what means that the properties /14.8/ lead uniquely to our solutions. They divide into three families, first of which contains new metrics, while the other two were known before, and we shall need them here. The second family solutions are:

$$ds^2 = H^{-2}[(dx^0)^2 + 2x^2\,dx^0\,dx^1 + \tfrac{1}{2}(x^2)^2(dx^1)^2] + \\ -[\varkappa\varrho H (x^2)^2]^{-1}(dx^2)^2 - 2\varkappa\varrho^{-1}H^3(dx^3)^2 \qquad /14.9/$$

where:

$$H = 1 + p/c^2\varrho \qquad /14.10/$$

$$\Lambda = \tfrac{1}{2}\varkappa(\varrho - p/c^2) \qquad /14.11/$$

ϱ = const > 0 , p = const ≥ 0, $\varkappa = 8\pi k/c^2$.

The third family solution is:

$$ds^2 = (dx^0)^2 + 2x^2 dx^0 dx^1 + x^2(x^2+1)(dx^1)^2 + + (\varkappa a x^2)^{-1} e^{x^2}(dx^2)^2 - \varkappa a^{-1} e^{x^2}(dx^3)^2 \qquad /14.12/$$

where:

$$\varrho = a\, e^{-x^2} \qquad /14.13/$$

a = const > 0, $\Lambda = p = 0$.

In the survey of exact solutions I make it my aim to do two things:

1. To show which solutions known up to now are special cases of ours, which are not, and what the difference is then.
2. To prove that for all these solutions our coordinates from part 1, in which the equations /14.1/ to /14.4/ are fulfilled, do exist.

The second object was not always possible to secure, as not all the authors gave enough information to construct the metric tensor and all the important scalars explicitly.

I was interested only in models of a pure perfect fluid /or dust/, without any electromagnetic fields, viscosity, heat conduction or anisotropy in pressure, because our considerations from part 1 cannot include such general situations.

In the following sections I will not explain, how to look for a transformation to the coordinates of /14.1/ - /14.4/. I give now the general idea of this procedure to make it clear to the reader /the example in section 4 could be helpful/. First, we pick such coordinate x^0 that $u^\alpha = H\delta^\alpha_0$ /if $p=0$, then $H = 1$ and this is just a comoving reference frame/. Then automatically $g_{00} = H^{-2}$. Next, we specialize the other three coordinates in such a way, that only one of the metric components g_{01}, g_{02}, g_{03} is different from zero, and we pick out x^1 so that just $g_{01} \neq 0$. Then we compute $H^2 \cdot g_{01}$ and choose the obtained function as the x^2-coordinate. Finally, we specialize x^3 so that $w^\alpha = \varsigma H^{-1}\delta^\alpha_3$ what is equivalent to $g = -\varsigma^{-2} H^{-2}$. The theorems and formulas of chapter 1 from part 1 ensure that this procedure can be carried out. We recall /see chapter 1e/ that once these coordinates are introduced, the properties /14.8/ are invariant.

The survey is based on the bibliography list from Synge's book [34] until 1925, and on the annals of "Physics Abstracts" for later papers.

1. LANCZOS' SOLUTION [35]

In the general case, with non-vanishing cosmological constant, this solution is of the form (1):

$$ds^2 = dt^2 - 2Cu\,d\varphi\,dt + [C^2(u^2-u) - \lambda(1-e^{-u})]\,d\varphi^2 + \\ -\frac{1}{4}e^{-u}[C^2u + \lambda(1-e^{-u})]^{-1}du^2 - e^{-u}dy^2 \qquad /14.14/$$

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(1) We have changed from the signature /+++-/, used by the author, to /+---/.

where C is an arbitrary constant, λ is the cosmological constant, and:

$$\rho = 2\lambda + 4C^2 e^{u} \qquad /14.15/$$

is the density of matter. We have p = 0, $u^\alpha = \delta^\alpha_0$,
$w^\alpha = 2Ce^u\delta^\alpha_3$, $(x^0, x^1, x^2, x^3) = (t, \psi, u, y)$.

Since for dust H = 1, the conditions /14.1/ and /14.6/ are fulfilled. To fulfil /14.2/+ /14.4/ and /14.7/ the following transformation is needed:

$$\begin{aligned} \psi &= C^{-1}x^1 \\ u &= -x^2 \\ y &= C[\lambda e^{x^2} + 2C^2]^{-1}\, x^3 \end{aligned} \qquad /14.16/$$

Then /14.14/ changes to:

$$\begin{aligned} ds^2 = &(dx^0)^2 + 2x^2 dx^0 dx^1 + [x^2(x^2+1) - \lambda C^{-2}(1-e^{x^2})](dx^1)^2 + \\ &- \left[\frac{1}{4}\,\frac{e^{x^2}}{\lambda(1-e^{x^2}) - C^2x^2} + \frac{\lambda^2C^2e^{3x^2}(x^3)^2}{(\lambda e^{x^2}+2C^2)^4}\right](dx^2)^2 + \\ &+ 2\frac{\lambda C^2 e^{2x^2}x^3}{(\lambda e^{x^2}+2C^2)^3}\, dx^2 dx^3 - \frac{C^2e^{x^2}}{(\lambda e^{x^2}+2C^2)^2}(dx^3)^2 \end{aligned} \qquad /14.17/$$

New we see why this metric could not appear among our solutions when $\lambda \neq 0$: it depends then on x^3, so the second of /14.8/ does not hold. However, when $\lambda = 0$ the metric /14.17/ is identical with our third family solution /14.12/-/14.13/, if only $4C^2$ = a./We should remember that Lanczos used such system of units in which $\varkappa$ = 1/.

The metric /14.14/ was very nicely discussed in [35], so that only a few questions /not to be answered at that time/

were left open. For the reasons which I do not understand, the paper of Lanczos fell into a complete oblivion, so that his solution has been at least twice rediscovered /see sections 2 and 5/ until S.C. Maitra /see section 7/ recalled it to the physical community in 1966.

2. VAN STOCKUM'S SOLUTION [36]

He gave the following metric(1):

$$ds^2 = c^2\,dt^2 - 2c\,a\,r^2\,dt\,d\varphi + \qquad /14.18/$$
$$- r^2(1-a^2r^2)\,d\varphi^2 - e^{-a^2r^2}(dr^2+dz^2)$$

This solution is not merely identical with that of Lanczos [35] /in the case $\lambda = 0$/, but even represented in the same coordinate system which Lanczos used in the first part of his paper. The metric /14.18/ reduces to /14.12/ after the following transformation:

$$ct = x^0$$
$$\varphi = a x^1 \qquad /14.19/$$
$$r = a^{-1}(-x^2)^{1/2}$$
$$z = a^{-1}x^3$$

The "a" above should be identified with $a^{1/2}$ from /14.12/. Here again $\varkappa = 1$.

Van Stockum did not quote Lanczos, although he quoted quite a few times the paper of Lewis [37] where the reference to Lanczos was given.

The so called "general van Stockum's solution [36] " depends on one function which is to be determined from a partial differential equation. We shall not consider it here.

3. GÖDEL'S SOLUTION [38]

In [38] this solution is given in the form:

$$ds^2 = a^2\left(dx_0^2 - dx_1^2 + \frac{1}{2} e^{2x_1} dx_2^2 - dx_3^2 + 2e^{x_1} dx_0 dx_3\right) \qquad /14.20/$$

where the velocity field, the density of matter and the cosmological constant are:

$$u^\alpha = a\,\delta_0^\alpha$$
$$\rho = (8\pi k a^2)^{-1} = \text{const} \qquad /14.21/$$
$$\lambda = -(2a^2)^{-1} = -4\pi k \rho$$

Gödel used the system of units in which c = 1. The second formula of /14.21/ can be thus rewritten as:

$$a = (\varkappa \rho)^{-1/2}, \quad \varkappa = 8\pi k/c^2 \qquad /14.22/$$

Now we execute the following coordinate transformation:

$$x_0 = a^{-1}x^0$$
$$x_1 = \ln x^2 \qquad /14.23/$$
$$x_2 = a^{-1}x^1$$
$$x_3 = \sqrt{2}\,\varkappa x^3$$

We find that the resulting metric is /14.9/+/14.11/ with $p = 0$ and $\Lambda = -\lambda$.

Gödel's solution /14.20/-/14.21/ was not the first model of rotating matter, although the author claimed so. However, his statement has been repeated uncritically for the next 17 years, until S.C. Maitra reminded Lanczos and van Stockum. Ocasionally it happens until today that some people refer to Gödel as the first author of a model of rotating Universe.

4. NARLIKAR'S METRIC [39]

The author gives no new solutions, but proposes a general metric which would describe the rotating Universe filled with dust. We show that this metric may be put in the form /14.1/ + /14.4/.

It is assumed that matter rotates around an axis with vorticity vector parallel to that axis. Therefore every particle moves on the surface of a cylinder, but the axial symmetry of motion is not assumed. The author demands further that the section of the cylinder with a surface orthogonal to the generator is a two-dimensional space of constant curvature. These assumptions imply the following form for the metric:

$$ds^2 = dt^2 + 2f(1-kr^2)^{-1/2} dr\,dt + 2gr\,d\varphi\,dt + \qquad /14.24/$$
$$- a^2(t)\left[(1-kr^2)^{-1} dr^2 + r^2 d\varphi^2\right] - s^2(t)\,dz^2$$

where a and s are functions of t; f and g are functions of r and φ. The constant $k = 0, \pm 1$ is the curvature of the cylinder's cross-section. The vorticity vector has the direction of the z-axis.

Since $u^\alpha = \delta^\alpha_0$, the condition /14.6/ is fulfilled. But the comoving reference frame is defined exact to the transformations:

$$x^0 = x^{0'} + F^0(x^{1'}, x^{2'}, x^{3'})$$

$$x^i = F^i(x^{1'}, x^{2'}, x^{3'}) \quad , \quad i = 1, 2, 3.$$

where F^0, F^i are arbitrary functions of three variables. Therefore we will not destroy /14.6/ if we execute the transformation:

$$t = \tau - J(r, \varphi) \qquad /14.25/$$

where:

$$J(r, \varphi) \overset{\mathrm{def}}{=} \int (1 - kr^2)^{-1/2} f(r, \varphi)\, dr \qquad /14.26/$$

The metric /14.24/ changes now to:

$$ds^2 = d\tau^2 + 2\left(gr - \frac{\partial J}{\partial \varphi}\right) d\tau d\varphi - 2\frac{fgr}{(1-kr^2)^{1/2}} d\varphi\, dr + \qquad /14.27/$$

$$+ \left[\left(\frac{\partial J}{\partial \varphi}\right)^2 - 2gr\frac{\partial J}{\partial \varphi} - r^2 a^2(t)\right] d\varphi^2 - \frac{f^2 + a^2(t)}{1 - kr^2} dr^2 - s^2(t) dz^2$$

If /14.24/ is not static /what is assumed/, then the following transformation is nonsingular:

$$x^2 = gr - \frac{\partial J}{\partial \varphi} \qquad /14.28/$$

$$x^1 = \varphi$$

After /14.28/ is done, the metric /14.27/ obeys /14.1/ ÷ /14.3/ and /14.6/, so that only /14.4/ and /14.7/ remain.

In the comoving frame the equation of continuity assumes the form $(\sqrt{-g}\,\rho)_{,0} = 0$, hence:

$$g = -\rho^{-2} F(x^1, x^2, z) \qquad /14.29/$$

where F is an arbitrary function of three variables. But all the components of /14.27/ are independent of z, so in our case $F = F(x^1, x^2)$. This enables us to execute the last transformation:

$$z = \frac{x^3}{F(x^1, x^2)} = \frac{x^3}{\varrho\sqrt{-g}} \qquad /14.30/$$

which does not destroy the previously assured conditions. Now all the equations /14.1/ ÷ /14.4/, /14.6/ and /14.7/ are fulfilled.

5. WRIGHT'S SOLUTION [40]

The author obtained the Gödel's metric [38] and the Lanczos' metric [35]. Not knowing about Lanczos and van Stockum however, he ascribed himself the priority in discovering this unlucky metric. S.C. Maitra /see section 7/ and G.F.R. Ellis /see section 10/ already recognized this fact. Therefore we only show how to transform Wright's coordinates into Lanczos' coordinates.

J.P. Wright represented his metric in the form(2):

$$\begin{aligned} ds^2 = {} & (dx^0)^2 - 4y^2 e^{-2y^2}\left(D + \tfrac{1}{2}A^2y^2 - \lambda e^{-2y^2}\right)^{-1} dy^2 + \\ & + 2\left(\pm\frac{A}{B}y^2 + C\right) dx^0 dx^2 + \\ & + \left[-\frac{1}{2}\left(\frac{A}{B}\right)^2 y^2 + \frac{\lambda}{B^2}e^{-2y^2} - \frac{D}{B^2} + \left(\pm\frac{A}{B}y^2 + C\right)^2\right](dx^2)^2 + \\ & - e^{-2y^2}(dx^3)^2 \end{aligned} \qquad /14.31/$$

The density of matter is equal to:

$$\varrho = \varkappa^{-1}\left(A^2 e^{2y^2} + 2\lambda\right) \qquad /14.32/$$

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(2) There is a misprint in the paper [40]. Inside the last term in square brackets there should be a double sign before $\frac{A}{B}$, but the lower is not printed.

We notice first that the coordinate transformation $x^0 = x^{0'} - Cx^2$ yields the same result as if $C = 0$, so with no loss in generality we can assume $C = 0$. Next, we see that the reflection $x^2 \longrightarrow -x^2$ changes + before $\frac{A}{B}$ to −, so the double sign is not significant, and we shall deal with + only. Finally, the transformation $x^2 = B\, x^{2'}$ yields $B = 1$, so we assume $B = 1$.

Now, we perform the transformation:

$$x^0 = x^{0'} - \frac{1}{2} AE\, e^{\frac{1}{2}E} x^{1'}$$
$$y = \left[\frac{1}{2}\left(x^{2'} + E\right)\right]^{1/2} \qquad /14.33/$$
$$x^2 = e^{\frac{1}{2}E} x^{1'}$$
$$x^3 = e^{\frac{1}{2}E} x^{3'}$$

where E is a solution of:

$$\left(D + \frac{1}{4} A^2 E\right) e^E = \lambda \qquad /14.34/$$

Such E exists if only $\lambda \geqslant 0$, what must be the case for otherwise ϱ would not be positive in some region of $x^{2'}$. If we identify then $\frac{1}{2} Ae^{\frac{1}{2}E}$ with C, we obtain /14.14/ where $\left(x^{0'}, x^{1'}, x^{2'}, x^{3'}\right) = \left(t, \psi, u, y\right)$.

6. THE SOLUTIONS OF OZSVÁTH AND SCHÜCKING [41], [42] AND OZSVÁTH [43]

The "finite rotating Universe" discussed in [41] and [42] is a special case of the solutions presented in [43], so we shall not discuss it separately.

The author obtained six different solutions, all describing dust. We investigate only the first one in detail. It is of the form:

$$ds^2 = a^{-2}[(1-\nu_0^2)e^{-2\lambda_0 x^3}(dx^0)^2 + 2(1-\nu_0\nu_1)e^{-x^3}dx^0dx^1 + \\ + (1-\nu_1^2)e^{-2\lambda_1 x^3}(dx^1)^2 - e^{-2\lambda_2 x^3}(dx^2)^2 - (dx^3)^2] \qquad /14.35/$$

where a and s are constants, $\frac{1}{2} \leq s^2 \leq 2$, and:

$$\beta = [1 + 2s^2(1-s^2)(3-s^2)]^{1/2}$$
$$\left.\begin{matrix}\lambda_0\\ \lambda_1\end{matrix}\right\} = \frac{1}{2}(1 \mp \beta)$$
$$\lambda_2 = 1 - s^2 \qquad /14.36/$$
$$\nu_i = \frac{\sqrt{2}}{s(3-s^2)}\lambda_i, \quad i = 1,2.$$

The velocity field and the density of matter are:

$$u_\alpha = a^{-1}(e^{-2\lambda_0 x^3}, e^{-\lambda_1 x^3}, 0, 0) \qquad /14.37/$$

$$\varrho = \varkappa^{-1}a^2(2-s^2)(2s^2-1) = \mathrm{const} \qquad /14.38/$$

Since u_α has a nonzero shear tensor, we see at once that /14.35/ is not contained in our solutions.

The vorticity field is tangent to the x^2-line. We start with the transformation:

$$x^0 = u^0 x^{0'}$$
$$x^1 = a x^{1'} + u^1 x^{0'} \qquad /14.39/$$
$$x^2 = x^{3'}$$
$$x^3 = -\lambda_1^{-1}\ln x^{2'}$$

where $u^0 = a\nu_1(\nu_1-\nu_0)^{-1}e^{-(\lambda_1-1)x^3}$, $u^1 = -a\nu_0(\nu_1-\nu_0)^{-1}e^{-(\lambda_0-1)x^3}$ are the contravariant components of the velocity field /14.37/. Then /14.35/ changes to:

$$ds^2 = (dx^{0'})^2 + 2x^{2'}dx^{0'}dx^{1'} + (1-\nu_1^2)(x^{2'})(dx^{1'})^2 + \\ + 2\frac{1-s^2}{3-s^2}x^{0'}dx^{1'}dx^{2'} - \frac{2+a^2s^2(1-s^2)^2(x^{0'})^2}{2a^2\lambda_1^2(x^{2'})^2}(dx^{2'}) + \qquad /14.40/ \\ - a^{-2}(x^{2'})^{2\lambda_2/\lambda_1}(dx^{3'})^2$$

Now /14.1/ ÷ /14.3/ and /14.6/ are fulfilled. In order to fulfil /14.4/ and /14.7/ the determinant of /14.40/ should be equal to $(-\varrho^{-2})$, while actually it is:

$$g' = -\frac{2(x^{2'})^{2\lambda_2/\lambda_1}}{a^4 s^2 (3-s^2)^2} \qquad /14.41/$$

Therefore we perform another transformation:

$$x^{3'} = x^{3''} / \varrho\sqrt{-g'} \qquad /14.42/$$

where ϱ and g' are given by /14.38/ and /14.41/ respectively. After the transformation we obtain:

$$\begin{aligned} ds^2 = {} & (dx^{0'})^2 + 2x^{2'}dx^{0'}dx^{1'} + (1-\nu_1^2)(x^{2'})^2(dx^{1'})^2 + \\ & -\frac{1}{\lambda_1^2 (x^{2'})^2}\left[\frac{1}{a^2} + \frac{1}{2}s^2(1-s^2)^2(x^{0'})^2 + \frac{\varkappa^2\lambda_2^2 s^2(3-s^2)^2(x^{3''})^2}{2a^2(2-s^2)^2(2s^2-1)^2}\right](dx^{2'})^2 + \\ & + 2\frac{1-s^2}{3-s^2}x^{0'}dx^{1'}dx^{2'} + \frac{\varkappa^2\lambda_2 s^2(3-s^2)^2 x^{3''}}{a^2\lambda_1(2-s^2)^2(2s^2-1)^2 x^{2'}}dx^{2'}dx^{3''} + \qquad /14.43/ \\ & -\frac{\varkappa^2 s^2(3-s^2)^2}{2a^2(2-s^2)^2(2s^2-1)^2}(dx^{3''})^2 \end{aligned}$$

This metric obeys all the conditions /14.1/ ÷ /14.4/, /14.6/ and /14.7/, but fulfils none of the assumptions /14.8/. This is why it could not appear among our solutions.

In the special case s = 1 we have $\lambda_1 = 1$, $\lambda_2 = 0$, and /14.43/ reduces to the Gödel's solution, i.e. /14.9/ ÷ /14.11/ with p = 0 /see section 3/.

Proceeding the same way as above, we can introduce the coordinates of /14.1/ ÷ /14.4/ for four of the other five solutions from [43]. The "finite rotating Universe" is a special case of them. In all the four cases it turns out that after the metric is put in the form /14.1/ ÷ /14.4/, its components depend both on x^0 and x^3, so they do not fulfil /14.8/, and could not appear among our solutions.

However, for the metric /6.14/ from [43] this procedure fails. This solution contains a hidden mistake. The tetrad $e^a{}_j$, given by /6.13/ in [43], fulfils $\mathrm{Diag}[+1,-1,-1,-1] = e_a{}^k g_{kl} e_b{}^l$, but the velocity vector $u_j = e^0{}_j$ does not obey the equations of motion. It turns out that $(u_{i;j}-u_{j;i})u^j = \frac{1}{2}\left[(b^2+1)/(b^2-1)\right]^2 \delta^3_i$ although it should be $(u_{i;j}-u_{j;i})u^j=0$. To point out clearly where the mistake was made, one should go deep into the material of [43], so I will not do it here.

7. MAITRA'S SOLUTION [44]

This solution is different from ours because it has nonzero shear tensor. It has the form:

$$ds^2 = dt^2 - e^{2\psi}(dr^2+dz^2) - (r^2-m^2)d\varphi^2 - 2m\,d\varphi\,dt \qquad /14.44/$$

where:

$$\psi = -\frac{1}{4x^2}\left(\sqrt{1+x^2}-1\right) + \frac{1}{8} - \frac{1}{4}\ln\frac{1}{2}\left(\sqrt{1+x^2}+1\right)$$

$$m = \frac{1}{2}a\left[\sqrt{1+x^2}-1-\ln\frac{1}{2}\left(\sqrt{1+x^2}+1\right)\right] \qquad /14.45/$$

$$r = \frac{1}{2}ax \quad , \quad a = \mathrm{const}$$

The pressure p = 0, while the velocity field and the density of matter are given by:

$$v^t = \frac{r-mm'}{r\sqrt{1-m'^2}}$$

$$v^\varphi = -\frac{m'}{r\sqrt{1-m'^2}} \qquad /14.46/$$

$$\varrho = \frac{4\cdot e^{-2\psi}}{\varkappa a^2}\cdot\frac{\sqrt{1+x^2}-1}{x^4\sqrt{1+x^2}}$$

where $m' \stackrel{\mathrm{def}}{=} \frac{dm}{dr}$. We perform the transformation to the comoving frame of reference:

$$t = v^t x^0$$

$$\varphi = x^1 + v^\varphi x^0 \qquad /14.47/$$

The resulting metric is:

$$ds^2 = (dx^0)^2 + 2\frac{(r m' - m)\sqrt{r}}{\sqrt{a m'}} dx^0 dx^1 - (r^2 - m^2)(dx^1)^2 +$$

$$+ 2\frac{\sqrt{r}\, x^0}{\sqrt{1+x^2}} \left(\frac{m'}{a}\right)^{3/2} (r - m m') dx^1 dr + \qquad /14.48/$$

$$- \left[e^{2\psi} + \frac{m'^4 (x^0)^4}{a^2(1+x^2)} \right] dr^2 - e^{2\psi} dz^2$$

The equations /14.1/, /14.3/ and /14.6/ are now fulfilled. To secure /14.2/ we perform the next transformation; which changes r to a function $x^2(r)$:

$$(r m' - m)(r / a m')^{1/2} = x^2 \qquad /14.49/$$

Although the metric depends on x^0, its determinant g does not. Therefore, without destroying the condition /14.3/, we can perform the third transformation:

$$z = x^{3'} / g\sqrt{-g} \qquad /14.50/$$

After /14.50/ is done, the Maitra's solution obeys all the conditions /14.1/ + /14.4/.

It depends then both on x^0 and x^3; so it does not fulfil /14.8/, and that is why it could not appear in our third family.

8. THE SOLUTION OF RAVAL AND VAIDYA [45]

The authors looked for generalizations of the Gödel solution to the case of non-perfect fluid, in the sense that the distribution of pressure was anisotropic. The principal values of pressure in the direction of vortex and in the orthogonal directions were different. There is only one special case when the pressure is isotropic, and only this one may be compared with our solutions:

$$ds^2 = A^2[(dx^0)^2 + 2e^{x^1}dx^0dx^2 + \tfrac{1}{2}e^{2x^1}(dx^2)^2 + - (dx^1)^2 - (dx^3)^2 + 2\sqrt{B/(1-B)}\,dx^0dx^1] \qquad /14.51/$$

where A, B = const, $0 \leq B < \frac{1}{2}$, and:

$$\varkappa\varrho = \frac{\varkappa p}{c^2} = \frac{1-B}{1-2B}\cdot\frac{1}{2A^2} \qquad /14.52/$$

If B = 0 and $\varkappa p/c^2$ is identified with $(-\lambda)$, then /14.51/ becomes identical with Gödel's solution /14.20/ (Note: if $\varkappa p/c^2$ is identified with $-\lambda$, then ϱ should be interpreted as $(\varrho + \lambda/\varkappa)$).

We show that when B $\neq$ 0 the metric /14.51/ is identical with /14.9/ + /14.11/, i.e. our second family. First, we reinterpret the constants $\varkappa\varrho$ and $\varkappa p/c^2$: we write $(\varkappa\varrho - \Lambda)$ instead of $\varkappa\varrho$, and $(\varkappa p/c^2 + \Lambda)$ instead of $\varkappa p/c^2$. It is easy to see that this may always be done, as the field equations do not differentiate these situations. Then:

$$\Lambda = \frac{1}{2}\varkappa(\varrho - p/c^2) \qquad /14.53/$$

in agreement with /14.11/, and:

$$\frac{1-B}{1-2B}\cdot\frac{1}{2A^2} = \frac{1}{2}\varkappa\left(\varrho + \frac{p}{c^2}\right) = \frac{1}{2}\varkappa\varrho H \qquad /14.54/$$

Now we perform the coordinate transformation:

$$\begin{aligned} x^0 &= (AH)^{-1}x^{0'} + [B/(1-B)]^{1/2}\ln x^{2'} \\ x^1 &= \ln x^{2'} \\ x^2 &= (AH)^{-1}x^{1'} + 2[B/(1-B)]^{1/2}\cdot 1/x^{2'} \\ x^3 &= \varkappa H^2[2(1-2B)/(1-B)]^{1/2}\,x^{3'} \end{aligned} \qquad /14.55/$$

As a result, we obtain the metric /14.9/.

9. THE PAPER OF TRÜMPER [46] AND OTHERS RELATED

There are no new solutions in the paper of Trümper, but it is closely related to our work. The author considered a rotating perfect fluid which is stationary and axially symmetric, with the velocity vector colinear to the timelike Killing vector. From these properties he deduced the following form of the metric:

$$ds^2 = e^{2U}(dt + A\,d\varphi)^2 - e^{-2U}\left[F^2 d\varphi^2 + e^{2k}(dr^2 + dz^2)\right] \qquad /14.56/$$

where (t, φ, r, z) are cylindrical coordinates, and U,A, F,k are functions of r and z only. The author wrote down the field equations for /14.56/, but did not try to integrate them. He proved that if $p_{,\alpha} \neq 0$, then the density of matter is either constant or the function of pressure p.

We see that if somebody integrated the equations written by Trümper, he would obtain a generalization of our first family solutions. If the functions U,A,F,k were assumed to be independent of z, then the solutions would reduce to the metrics given in chapter 5 from [1], transformed to another coordinate system. It is easy to see, that the metrics /5.12/ from [1] are of the form /14.56/: if a new coordinate r is introduced by $(W\varrho H)^{-1}\left(\frac{dx^2}{dr}\right)^2 = G\varrho^{-1}H^3$ then $e^{2U(r)} = H^{-2}(x^2(r))$, $A(r) = x^2(r)$, $F^2(r) = W/GH^4\big|_{x^2 = x^2(r)}$, $e^{2k(r)} = -G\varrho^{-1}H^3\big|_{x^2 = x^2(r)}$.

There are other papers whose authors have written down the field equations for a rotating perfect fluid, but confined themselves only to more or less penetrating discussion, or had just other purpose than integration of the field equations. These papers can be found in the bibliography list under the numbers [47] ÷ [50] .

10. ELLIS' SOLUTIONS [51]

Some of the solutions considered in [51] describe a nonrotating portion of matter. Here we discuss only those with rotation. According to the author's classification these are: case I, and the classes A,B,C of shear-free solutions. All of them describe dust.

Case I is divided into the subcases Ia/ and Ib/. The subcase Ia/ is the Gödel's solution [38] , discussed in section 3. In the subcase Ib/ the solution is (1) :

$$ds^2 = (dx^0)^2 - (dx^1)^2 - Y^2(x^1)\left[(dx^2)^2 + t^2(x^2)(dx^3)^2\right] + \\ - y^2(x^2)\left[2\,dx^0 - y(x^2)\,dx^3\right]dx^3 \qquad /14.57/$$

where t and y are defined by differential equations with initial conditions:

$$\frac{d^2t}{d(x^2)^2} + Kt = 0\,,\; K=\text{const},\; t(0)=1,\; \frac{dt}{dx^2}(0)=0$$

$$\frac{dy}{dx^2} = -2ct(x^2)\,,\quad c=\text{const}\,,\; y(0)=0 \qquad /14.58/$$

The function $Y(x^1)$ is defined as follows:

$$Y(x^1)=\begin{cases} 1+2cx^1 & \text{when } \Lambda=0=K \\ K^{-1}[(\sqrt{c^2+K}+Kx^1)^2-c^2] & \text{when } \Lambda=0\neq K \\ (2\Lambda)^{-1}[\sqrt{K^2+4c^2\Lambda}\,\sin(2\sqrt{\Lambda}x^1)+K] & \\ \qquad\qquad \text{when } \Lambda>0 & \\ \frac{1}{2}e^{-\sqrt{-\Lambda}x^1}[(e^{2\sqrt{-\Lambda}x^1}+K/2\Lambda)^2+c^2/\Lambda] & \\ \qquad\qquad \text{when } \Lambda<0 & \end{cases} \qquad /14.59/$$

The metric /14.57/ fulfils the conditions /14.1/ and /14.6/ because its reference system is comoving. To fulfil /14.2/ and /14.3/ we perform the transformation:

$$\begin{aligned} x^1 &= x^{3'} \\ x^3 &= -x^{1'} \\ y(x^2) &= x^{2'} \end{aligned} \qquad /14.60/$$

We denote $x^2 = u(x^{2'})$ the function reciprocal to $x^{2'} = y(x^2)$, and find from /14.59/:

$$ds^2=(dx^0)^2+2x^{2'}dx^0dx^{1'}+[(x^{2'})^2-Y^2(x^{3'})t^2(u(x^{2'}))](dx^{1'})^2+$$
$$-(4c^2t^2)^{-1}Y^2(x^{3'})(dx^{2'})^2-(dx^{3'})^2 \qquad /14.61/$$

If y = const, then the transformation /14.60/ is singular, but then the matter does not rotate, and we need not take this case into account.

The density of matter /see [51] / depends only on $x^{3'}$. The determinant of /14.61/ is equal to:

$$g' = -Y^4(x^{3'})/4c^2 \qquad /14.62/$$

Thus we perform the transformation:

$$x^{3''} = (2c)^{-1}\int \varrho(x^{3'})Y^2(x^{3'})\,dx^{3'} \qquad /14.63/$$

which secures /14.4/ without destroying /14.1/ + /14.3/. Since the vorticity vector is tangent to x^3 - lines, we see that the second of /14.8/ is broken, and this metric does not belong to our third family.

Now we pass on to shear - free solutions. The class A contains case I considered above, and a new solution which is not explicitly given in [51], so we will not try to put it in the form /14.1/ + /14.4/. In this solution the derivative of the density of matter in the direction of vortex is different from zero, so the second of /14.8/ is certainly broken.

The class B contains the solutions of Lanczos [35] and Gödel [38] discussed previously.

The class C divides into the subclasses Ci, Cii, Ciii. The subclass Ci is identical with the class A. In the class Cii the solutions are not explicitly given, but they are seen to be different from ours because the scalar of rotation $\omega = \frac{1}{2}(-g_{\alpha\beta}w^{\alpha}w^{\beta})^{1/2}$ has nonzero derivative in the direction of vortex, and /14.8/ is broken. The metric of the class Ciii is given by (1):

$$\begin{aligned} ds^2 = &(dx^0)^2 - A^2(x^3)B^2(x^3)(dx^1)^2 - A^2(x^3)(dx^2)^2 + \\ &+ 4x^2\,dx^0dx^3 + 2cx^1B^2(x^3)\,dx^2dx^3 + \\ &+ \{4(x^2)^2 - B^2(x^3)[1 + c^2(x^1)^2B^2(x^3)/A^2(x^3)]\}(dx^3)^2 \end{aligned} \qquad /14.64/$$

where c = const while A and B are defined by some differential equations. Proceeding exactly the same way, as in the case of /14.57/ we can put /14.64/ in the form /14.1/ + /14.4/, but then the second of /14.8/ appears to be broken.

In the coordinate system of /14.64/ the verticity vector is tangent to the x^1-lines.
Therefore one might expect that /14.64/ fulfils /14.8/ when $c = 0$. We prove that it is not so. Let us perform the transformation $x^1 = x^{3'}$, $x^2 = \frac{1}{2} x^{2'}$, $x^3 = x^{1'}$. Then /14.64/ with $c = 0$ becomes:

$$ds^2 = (dx^0)^2 + 2x^{2'} dx^0 dx^{1'} + [(x^{2'})^2 - B^2(x^{1'})](dx^{1'})^2 + \\ - \frac{1}{4} A^2(x^{1'})(dx^{2'})^2 - A^2(x^{1'}) B^2(x^{1'})(dx^{3'})^2 \qquad /14.65/$$

This metric obeys /14.1/ ÷ /14.3/, but not /14.4/. To make it obey /14.4/ we must perform the next transformation:

$$x^{3'} = 2x^{3''} / \varrho(x^{1'}) A^2(x^{1'}) B^2(x^{1'}) \qquad /14.66/$$

where $\varrho(x^{1'})$ is the density of matter /see [51] /.
This transformation does not introduce $x^{3''}$ into /14.65/ only when $\varrho A^2 B^2$ = const. We know from [51] that this is true for the metrics of Lanczos [35] and Gödel [38] only.
Consequently, none of the Ellis' solutions belong to any of our families in [1].

11. THE SOLUTION OF STEWART AND ELLIS [52]

The authors considered the Einstein field equations for a fluid with nonzero anisotropic pressure, nonzero viscosity, electrically charged and moving in an electromagnetic field. We pick out only those special cases, in which the pressure is isotropic, the viscosity, eletric charge and electromagnetic field vanish, while the vorticity is different from zero. They are contained in case I according to authors' classification. It is divided into the classes Ia, Ib, Ic and Id. In the class Ia, when the electromagnetic field vanishes, the solution describes matter with constant density and constant pressure, and is identical with the solution of Raval and Vaidya /14.51/ /though the authors do not give this reference/. The solution of class Ic does not exist without the electromagnetic field. In the class Id no solution is given explicitly, and therefore we are not able to discuss it.

The information given about the class Ib is too little to discuss the conditions /14.1/ + /14.4/. However, an argument showing that our first family does not belong to them, may be given. In all the solutions from [52] only one direction orthogonal to the velocity field u^α may be distinguished. Thus the vectors of vorticity and acceleration, if both nonzero, must be colinear, and consequently

$$g_{\alpha\beta} w^\alpha \dot{u}^\beta = \Phi(x)\,\omega\,\dot{u} \qquad /14.67/$$

where $\Phi(x) \neq 0$ is a scalar function, $\omega \stackrel{\text{def}}{=} \frac{1}{2}(-w_\alpha w^\alpha)^{1/2}$, $\dot{u} \stackrel{\text{def}}{=} (-\dot{u}_\alpha \dot{u}^\alpha)^{1/2}$.

However, the formulas /14.7/ and /12.2/ from 2 imply that in our case:

$$g_{\alpha\beta} w^\alpha \dot{u}^\beta = 0 \qquad /14.68/$$

The formulas /14.67/ and /14.68/ are consistent when $\omega = 0$, which is excluded by assumption, or $\dot{u} = 0$, what means $\dot{u}^\alpha = 0$. We know from [2] that $\dot{u}^\alpha = 0$ means

H = const, see chapter 12. Therefore /14.67/ holds only for the second and third family solutions.

12. WAHLQUIST'S SOLUTION [53]

The author found a very complicated solution which describes, in a special case, a spatially limited portion of a rotating perfect fluid. We shall confine our-selves to this special case only.

The formulas for the velocity field, vorticity vector and density of matter are not given in [53]. The metric is too complicated to compute these quantities off-hand, and therefore we shall not introduce the coordinates of /14.1/ + /14.4/. We just give the argument that Wahlquist's solution is different from our first family solutions.

In our first family the relation /14.68/ holds. The same relation for Wahlquist's solution implies $g_r(\xi,\zeta)\cdot g_i(\xi,\zeta)=0$, where the functions g_r and g_i are:

$$g_r+ig_i \overset{\text{def}}{=} \frac{k}{r_0}\left[\cot\left(\frac{k\lambda}{r_0}\right)-\frac{k\lambda}{r_0}\cdot\frac{1}{\sin^2\left(\frac{k\lambda}{r_0}\right)}\right] \qquad /14.69/$$

with k, r_o = const. The complex variable λ is defined in terms of ξ and ζ in the following way:

$$\frac{k\lambda}{r_0} = \arcsin(k\zeta) + i\,\operatorname{arsh}(k\xi) \qquad /14.70/$$

It can be shown that each of the equations $g_r = 0$ and $g_i = 0$ implies $k = 0$. So $g_r \cdot g_i = 0$ implies $g_r = g_i = 0$. We know from [53] that then the vorticity vector vanishes. Consequently, Wahlquist's solution does not fulfil /14.68/

with $\omega \neq 0$, and certainly is different from our solutions.

13. WAINWRIGHT'S SOLUTIONS [54]

The author searched for such solutions of the Einstein field equations, whose Weyl tensor is algebraically special, the repeated principal null congruence being geodesic and shearfree. Thus only /14.9/, and the special metric discussed in chapter 10 of [2], might belong to this class because all the other our metrics are of Petrov type I.

The solution /14.9/, i.e. that of Raval and Vaidya [45], does really belong to these solutions /the author, however, identifies it with the Gödel solution [38] /, while the other does not: its repeated principal null congruence has nonzero shear.

Only one new solution is explicitly given in [54]:

$$ds^2 = -\frac{3}{4(x^1)^3}(dx^1)^2 + \frac{15}{16(x^1)^3}(dx^2)^2 - \frac{3}{2(x^1)^2}dx^2dx^3 + \\ -\frac{3}{2x^1}dx^2dx^4 + 2dx^3dx^4 - x^1(dx^4)^2 \qquad /14.71/$$

where the pressure p, the energy-density A, the density of matter ϱ and the velocity field u^α are given by:

$$p = x^1$$
$$A = x^1$$
$$\varrho = (2/Dc^2)\sqrt{x^1} \qquad /14.72/$$
$$u^\alpha = \sqrt{x^1}\,\delta^\alpha_3 + (1/\sqrt{x^1})\,\delta^\alpha_4$$

D is an arbitrary constant and c is the velocity of light.

Hence, as $A + p = \varrho c^2 H$, we have:

$$H = D\sqrt{x^1} \qquad /14.73/$$

We shall put /14.71/ in the form /14.1/ ÷ /14.4/ by the following transformation:

$$\begin{aligned} x^1 &= 3/(2x^{2'}) \\ x^2 &= -x^{1'}/D \\ x^3 &= x^{0'}/D \\ x^4 &= (2/3D)\, x^{0'} x^{2'} - \frac{2}{3} c^2 D^2 (x^{2'})^{3/2} x^{3'} \end{aligned} \qquad /14.74/$$

The metric /14.71/ changes to:

$$\begin{aligned} ds^2 = {} & \frac{2x^{2'}}{3D^2}(dx^{0'})^2 + \frac{4x^{2'}}{3D^2} x^{2'} dx^{0'} dx^{1'} + \frac{5(x^{2'})^3}{18D^2}(dx^{1'})^2 + \\ & + \left[\frac{2x^{0'}}{3D^2} - c^2 D\sqrt{x^{2'}}\, x^{3'}\right] x^{2'} dx^{1'} dx^{2'} - \frac{2}{3} c^2 D (x^{2'})^{5/2} dx^{1'} dx^{3'} + \\ & - \left[1 - \left(\frac{2x^{0'}}{3D} - c^2 D^2 \sqrt{x^{2'}}\, x^{3'}\right)^2\right] \frac{(dx^{2'})^2}{2x^{2'}} + \\ & - \frac{4}{3} c^2 D^2 (x^{2'})^{3/2} \left(\frac{2x^{0'}}{3D} - c^2 D^2 \sqrt{x^{2'}}\, x^{3'}\right) dx^{2'} dx^{3'} - \frac{2}{3} c^4 D^4 (x^{2'})^2 (dx^{3'})^2 \end{aligned} \qquad /14.75/$$

The conditions /14.1/ ÷ /14.4/ are fulfilled. This metric could not appear in our paper because it does not fulfil both assumptions /14.8/.

14. NEW SOLUTIONS OF OZSVÁTH [55]

In this paper the author deals with the solutions of the Einstein field equations for dust, belonging to classes II and III according to the terminology of Farnsworth and Kerr [56] . They are different from ours because they depend on the time coordinate in the comoving frame of reference, and thus do not fulfil the first of /14.8/. We put both solutions in the form /14.1/ - /14.4/.

The class II solution is given by:

$$ds^2 = (dx^0)^2 + 2dx^0\left[\frac{\sqrt{2}\,Rk^2 e^{x^1}}{(4k^2+1)^{1/2}}dx^2 + \frac{R}{\sqrt{2}}\left(\frac{1-2k^2}{4k^2-1}\right)^{1/2}dx^3\right] + \qquad /14.76/$$

$$+ \frac{1}{4}R^2\Big[(-1+k\cos 2x^0)(dx^1)^2 + 2k\sin 2x^0\,dx^1dx^2 +$$

$$+ k(2k-\cos 2x^0)e^{2x^1}(dx^2)^2 + 2\sqrt{1-2k^2}\,e^{x^1}dx^2dx^3 - (dx^3)^2\Big]$$

where R, k = const, R > 0, $\frac{1}{2} < |k| \leq \frac{1}{\sqrt{2}}$, and the density of matter is:

$$\rho = \frac{2}{\varkappa R^2}\cdot\frac{4k^2-1}{1-k^2} = \mathrm{const} \qquad /14.77/$$

Note: in [55] there is probably a misprint. The term $2\sqrt{1-2k^2}\;e^{x^1}\cdot dx^2\,dx^3$ is written there with the sign "minus". There are many reasons to get convinced that it should have the sign "plus", as in /14.76/.

The metric /14.76/ is represented in the comoving frame, so /14.1/ and /14.6/ are fulfilled. To fulfil /14.2/ and /14.3/ we perform the transformation:

$$\begin{aligned} x^0 &= x^{0'} - \frac{R}{\sqrt{2}}\left(\frac{1-2k^2}{4k^2-1}\right)^{1/2}x^3 \\ x^1 &= \ln x^{2'} \\ x^2 &= \frac{(4k^2-1)^{1/2}}{\sqrt{2}\,Rk^2}x^{1'} \end{aligned} \qquad /14.78/$$

The result is /watch carefully the difference between x^0 and $x^{0'}$ because both are present/:

$$ds^2 = (dx^{0'})^2 + 2x^{2'}dx^{0'}dx^{1'} + \frac{4k^2-1}{8k^3}(x^{2'})^2(2k-\cos 2x^0)(dx^{1'})^2 +$$

$$+ \frac{R(4k^2-1)^{1/2}}{2\sqrt{2}\,k}\sin 2x^0\,dx^{1'}dx^{2'} + \frac{R^2}{4}(-1+k\cos 2x^0)\frac{(dx^{2'})^2}{(x^{2'})^2} + \qquad /14.79/$$

$$- \frac{R}{2k^2}\left(\frac{1}{2}\cdot\frac{1-2k^2}{4k^2-1}\right)^{1/2}x^{2'}dx^{1'}dx^3 - \frac{1}{4}\frac{R^2}{4k^2-1}(dx^3)^2$$

The determinant of /14.79/ is equal to $g' = -R^4(1-k^2)/2^7k^4 =$ $=$ const, so in order to fulfil /14.4/ it is enough to transform x^3 by a constant factor.

Now we consider the class III solution:

$$ds^2 = 2(\varkappa\varrho)^{-1/2}\Big[(dx^0)^2 + 2e^{x^1}dx^0dx^2 + \\ -\frac{1}{2}(1+s\cos 2x^0)(dx^1)^2 - se^{x^1}\sin 2x^0\, dx^1dx^2 + \\ +\frac{1}{2}e^{2x^1}(1+s\cos 2x^0)(dx^2)^2 - (dx^3)^2\Big] \qquad /14.80/$$

where ϱ = const is the density of matter, and $|s|<1$, $s=\text{const}$.

In the special case s = 0 the metric /14.80/ reduces to the Gödel solution [38] . The velocity field is $u^\alpha = (\varkappa\varrho/2)^{1/2}\delta^\alpha_0$.

We transform:

$$x^0 = (\varkappa\varrho/2)^{1/2}x^{0'} \\ x^1 = \ln x^{2'} \\ x^2 = (\varkappa\varrho/2)^{1/2}x^{1'} \\ x^3 = \varkappa(1-s^2)^{-1/2}x^{3'} \qquad /14.81/$$

After the transformation we obtain (watch again x^0 and $x^{0'}$):

$$ds^2 = (dx^{0'})^2 + 2x^{2'}dx^{0'}dx^{1'} + \frac{1}{2}(x^{2'})^2(1+s\cos 2x^0)(dx^{1'})^2 + \\ -\frac{\sqrt{2}\,s}{\sqrt{\varkappa\varrho}}\sin 2x^0\, dx^{1'}dx^{2'} - \frac{1+s\cos 2x^0}{\varkappa\varrho(x^{2'})^2}(dx^{2'})^2 + \\ -\frac{2\varkappa}{(1-s^2)\varrho}(dx^{3'})^2 \qquad /14.82/$$

Now all the conditions /14.1/ ÷ /14.4/ are fulfilled.

15. WOLFE'S PAPER [57]

This paper contains no new solutions, but it deserves a note here, as it is an attempt at evaluating the angular velocity and scalar of shear of the Universe on the basis of experimental data.

16. HERLT'S SOLUTION [58]

This solution describes a source of the well-known NUT solution. It is a portion of perfect fluid in stationary and axially symmetric rotation. The metric is(1):

$$ds^2 = -(dx^1)^2 - R_0^2 \sin^2\left(\frac{x^1}{R_0}+\gamma^2\right)\left[(dx^2)^2 + \sin^2 x^2\, d\varphi^2\right] + \\ + e^{2U}\left(dx^4 + 4\beta \sin^2\frac{x^2}{2}\, d\varphi\right)^2 \qquad /14.83/$$

where:

$$e^U = (1-y^2)^{1/2} u_0 \frac{\frac{1}{k} + \mathrm{sn}^2\left[\frac{\beta(\delta+u_0)}{2R_0}\,\mathrm{arth}\,y + \bar{c}, k\right]}{\frac{1}{k} - \mathrm{sn}^2\left[\frac{\beta(\delta+u_0)}{2R_0}\,\mathrm{arth}\,y + \bar{c}, k\right]}$$

$$y = \cos\left(\frac{x^1}{R_0} + \gamma^2\right) \qquad /14.84/$$

$$k = \frac{\delta - u_0}{\delta + u_0} < 1$$

$$\delta^2 = \frac{R_0^2}{\beta^2} - u_0^2 > 0$$

$u_0, R_0, \beta, \bar{c}, \gamma$ are constants, $\mathrm{sn}(z, k)$ is the Jacobi elliptic function. The components of the energy - momentum tensor $T_{\alpha\beta} = (\mu c^2 + p)\, u_\alpha u_\beta - p g_{\alpha\beta}$ are given by:

$$u^\alpha = e^{-U} \delta^\alpha_0$$

$$\mu c^2 = \frac{3}{R_0^2} + \frac{3\beta^2}{R_0^4 (1-y^2)^2} e^{2U} \qquad /14.85/$$

$$p + \frac{\mu c^2}{3} = \frac{2y^2}{R_0^2(1-y^2)} - \frac{2k\beta(\delta+u_0)y}{R_0^3(1-y^2)}\,\mathrm{sn}\left[\frac{\beta(\delta+u_0)}{R_0}\,\mathrm{arth}\,y + 2\bar{c}, k\right]$$

It is easy to verify that the derivatives of p and μ in the direction of vortex are different from zero, so the second of /14.8/ is broken and certainly Herlt's solution is different from ours.

Now we put /14.83/ in the form /14.1/ + /14.4/. First we transform:

$$x^4 = x^{0'}$$
$$x^1 = x^{3'}$$
$$x^2 = 2\,\mathrm{arc}\,\sin\left(\frac{1}{2}\sqrt{\frac{x^{2'}}{\beta}}\right) \qquad /14.86/$$
$$\varphi = x^{1'}$$

Then /14.83/ changes to /dropping primes/:

$$ds^2 = e^{2U(x^3)}(dx^0 + x^2 dx^1)^2 - R_0^2 \sin^2\left(\frac{x^3}{R_0} + \gamma^2\right) \times$$
$$\times\left[\frac{(dx^2)^2}{x^2(4\beta - x^2)} + \frac{4x^2(4\beta - x^2)}{\beta^2}(dx^1)^2\right] - (dx^3)^2 \qquad /14.87/$$

/From now on x^3 should be substituted for x^1 in the definition of y/.

Comparing /14.84/ and /14.85/ with /1.1/ and /1.12/ in [1] we see that /14.87/ fulfils /14.1/ + /14.3/ and /14.6/ if $H = e^{-U}$. The condition /14.4/ requires that the determinant of /14.87/ should be $g' = -(\mu + p/c^2)^{-2}$, while actually it is:

$$g = -4\beta^{-2} R_0^4 \sin^4\left(\frac{x^3}{R_0} + \gamma^2\right) e^{2U} \qquad /14.88/$$

Both g and g' depend only on x^3, so we can execute the transformation:

$$x^{3'} = 2\beta^{-1} R_0^2 \int \sin^2\left(\frac{x^3}{R_0} + \gamma^2\right) e^U (\mu + p/c^2)\, dx^3 \qquad /14.89/$$

Then /14.87/ fulfils also /14.4/.

17. BRAY'S SOLUTIONS [59] , [60]

In [59] two solutions are presented. The author considered the Einstein-Maxwell equations for a rotating perfect fluid in a magnetic field, starting from a metric very similar to that of Gödel [38] . Here we are interested only in these special cases, in which the magnetic field is zero. The first solution from [59] does not exist in this case, while the second one reduces either to flat metric or to the Gödel solution /according to specific values of constants/.

Other particular solutions of the same problem, four in number, were presented in [60].

In the absence of the magnetic field they are as follows: the first and third are flat or identical with Gödel's metric, the second one is just flat and the fourth one does not exist.

However, the notations used by the author are not explained in the paper, and the formulas for the hydrodynamic or electromagnetic quantities are not given, the only explicit results being the metrics themselves. Consequently, there is some possibility that my interpretation of Dr Bray's results is not true. If I have not mistaken, Bray's solutions in the absence of electromagnetic field are nothing but the Gödel solution. This fact was not indicated in the papers [59] and [60] , and therefore Bray's name is not marked with a star in the table XII /see below/.

18. CONFRONTATION OF MODELS OF ROTATING MATTER

The table XII visualizes how many times each metric appeared in independent papers, and which discoveries were involuntarily repeated. One "cell" contains the names of the discoverers of the same solution. The star at author's name means that he knew about his predecessors and did not expect to be first.

19. CONCLUDING REMARKS

It is not clear if our coordinates used in [1] and [2] are practically useful. They are a kind of "canonical coordinates", and are very helpful in comparing the ready solutions. However, there are no reasons to claim that they are generally best. Some simple solutions transformed to this system of coordinates became very complicated, compare e.g. /14.14/ with /14.17/, /14.35/ with /14.43/ or /14.71/ with /14.75/. It would not be easy to obtain these metrics working in our coordinates. May be they are well fitted just to the single problem we have considered in [1] and [2] . This question needs further investigation.

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TABLE XII: CONFRONTATION OF MODELS OF ROTATING MATTER

/in the state at the end of 1973/

<table>
<tr>
<td>Lanczos 1924 [35]
van Stockum 1937 [36]
Wright 1965 [40]
* Ellis 1967 [51]
* Krasiński 1973 [1]</td>
<td>Gödel 1949 [38]
* Wright 1965 [40]
* Ozsváth 1965 [43]
* Raval-Vaidya 1966 [45]
* Ellis 1967 [51]
* Wainwright 1970 [54]
* Ozsváth 1970 [55]
Bray 1972 [59] [60]
* Krasiński 1973 [1]</td>
<td>Ozsváth-Schücking 1962 [41] [42]
* Ozsváth 1965 [43]</td>
<td>Ozsváth 1965 [43]</td>
</tr>
<tr>
<td>Maitra 1966 [44]</td>
<td>Raval-Vaidya 1966 [45]
Stewart-Ellis 1968 [52]
Wainwright 1970 [54]
* Krasiński 1973 [1]</td>
<td>Ellis 1967 [51]
* Wainwright 1970 [54]</td>
<td>Stewart-Ellis 1968 [52]
* Wainwright 1970 [54]</td>
</tr>
<tr>
<td>Wahlquist 1968 [53]</td>
<td>Wainwright 1970 [54]</td>
<td>Ozsváth 1970 [55]</td>
<td>Herlt 1972 [58]</td>
<td>Krasiński 1973 [1] ,[2]</td>
</tr>
</table>

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